CAMBRIDGE MATHEMATICAL SERIES.

ELEMENTARY DYNAMICS

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BY

William Martin

W. M. BAKER, M.A.

FORMERLY SCHOLAR OF QUEENS' COLLEGE, CAMBRIDGE;
HEAD MASTER OF THE MILITARY AND CIVIL DEPARTMENT
AT CHELTENHAM COLLEGE.

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PREFACE TO THE SECOND EDITION.

The main points of this edition are:

1. The insertion of geometrical and graphical proofs and representations of the formulae

$$v = u + ft,\quad s = ut + \tfrac{1}{2}ft^2,\quad v^2 = u^2 + 2fs.$$

2. Velocity treated as the limiting value of $\dfrac{\Delta s}{\Delta t}$.

3. Acceleration treated as the limiting value of $\dfrac{\Delta v}{\Delta t}$.

4. The treatment of Work and Energy at an earlier stage.

5. The introduction of batches of Revision Papers at different stages.

6. Full geometrical and graphical treatment of Projectiles.

7. The introduction of worked-out problem of various types, e.g.

   Trail of smoke problems,
   Problems on Change of Velocity,
   Jerks of Inelastic Strings,
   Motion on a Moving Inclined Plane.

8. Motion of a Particle under a variable acceleration.

Altogether about sixty pages of new matter have been inserted.

The recommendations of the Committee of the Mathematical Association have been followed to a great extent, especially in leading up to the ideas of the Differential and Integral Calculus.

The student is assumed to possess a knowledge of elementary Trigonometry, Graphs, and in the later chapters, of the Parabola, and elementary Co-ordinate Geometry.

In some sets of examples, the harder problems are marked with an asterisk. An endeavour has been made to put all explanations in as simple a manner as possible for the sake of beginners, but at the same time, the range of work should be sufficient for Woolwich and Sandhurst candidates, and for students reading for Scholarships at the Universities.

With the kind permission of Mr J. M. Dyer, of Eton, I have inserted a good many examples from the collection (Mathematical Examples, now out of print) made by him and the late Mr R. Prowde Smith.

W. M. B.

July, 1905.

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CHAPTER I.

VELOCITY.

1. A POINT is said to be in motion when it changes its osition relative to surrounding objects.

Thus if the distance from a point B to a point A changes, ? is said to have motion relative to A; and we must notice hat this change may be a change in *direction* as well as a hange in *length*. At present we shall only consider a change n *length*.

2. DEF. *The* **velocity** *of a moving point is its rate of* *otion; or the rate at which it is changing its position.*

The velocity of a point is said to be uniform when the oint moves over equal distances in equal intervals of time, owever small those intervals may be.

Measurement of velocity. When uniform the **velocity f a point is measured by** the distance passed over in a nit of time; **when variable, it is measured,** at any nstant, by the distance which would be passed over in unit ime if the point moved during that unit of time with the ame velocity as at the instant under consideration.

Thus when we speak of a train as passing us at the rate f 30 miles an hour, we do not mean that it passes over 30 miles in each hour, but that it would do so if it continued o move at the same rate as when passing us.

3. The unit of time usually employed is one second.

The British standard unit of length is a yard, which is defined as the distance between the centres of two plugs in

a bronze bar kept in the Exchequer Office, the temperature of the bar being 62° Fahrenheit.

The unit generally employed by engineers in England is one-third of this distance, and is called a foot.

For scientific purposes the unit of length usually employed is a centimetre.

4. *The* **unit of velocity** *is the velocity of a point which moves uniformly over unit space in unit time.*

Thus a point which passes uniformly over one foot in one second has unit velocity when a foot and a second are taken as units of length and time respectively. Also when we speak of a point as having a velocity v, we mean that it passes over v units of length in unit time; and a point having an equal velocity *but in an opposite direction* has a velocity $-v$.

5. When a point moves with uniform velocity u, it passes over u units of length in unit time, therefore it passes over ut units of length in t units of time. Hence if s be the number of units of length passed over by a point moving with velocity u for t units of time,

$$\mathbf{s = ut.}$$

6. The *mean or average velocity* of a moving point during any interval (in which it is not moving uniformly), is the velocity of another point which, *moving uniformly*, passes over the same distance in the same time.

Thus if a point moves over 5 ft., 7 ft., 9 ft. respectively in three consecutive seconds, it moves over 21 ft. in 3 seconds, and its average or mean velocity is 7 ft. per second.

A *nautical mile* is generally taken to be 6080 feet. A *knot* is sometimes taken to mean a velocity of one nautical mile per hour.

7. Ex. i. *Express a velocity of 60 miles an hour in feet per second.*

$$60 \text{ miles an hour } = 60 \times 1760 \times 3 \text{ feet per hour}$$
$$= \frac{60 \times 1760 \times 3}{60 \times 60} \text{ feet per second}$$
$$= 88 \text{ feet per second.}$$

Ex. ii. *If the velocity of sound is* 1120 *ft. per sec., how long does it take to travel* 14 *miles?*

Let t secs. denote the time required.

$$14 \text{ miles} = 14 \times 1760 \times 3 \text{ feet},$$

$$\therefore\ 14 \times 1760 \times 3 = 1120t; \qquad [s = ut]$$

$$\therefore\ t = \frac{14 \times 1760 \times 3}{1120} \text{ secs.}$$

$$= 66 \text{ seconds.}$$

Ex. iii. *A point moves with a velocity represented by* 12, *and describes* 60 *feet in* 4 *seconds: find the unit of time if* 3 *feet be the unit of length.*

The point describes 60 ft. in 4 secs.

i.e. it describes 20 units of length in 4 secs.

$\therefore$ .... 1 unit .... $\frac{4}{20}$ sec.

$\therefore$ .... 12 units .... $\frac{4 \times 12}{20}$ secs. $= 2\frac{2}{5}$ secs.

But a velocity represented by 12 means the velocity of a point which passes over 12 units of length *in unit time.*

$\therefore$ $2\frac{2}{5}$ secs. is the unit of time.

EXAMPLES. I. *a*.

[In each of the following questions the velocity is uniform.]

1. Express a velocity of 40 miles per hour in feet per second.

2. Express a velocity of 44 ft. per sec. in miles per hour.

3. If I ride a bicycle at the rate of 15 miles per hour, what is my velocity in foot-second units?

4. A man runs at the rate of $\frac{88}{5}$ ft. per second; how long will he take to run a mile?

5. Taking the diameter of the earth as 8000 miles, what is the velocity in foot-second units of a man standing at the equator, in consequence of the daily revolution of the earth about its axis? ($\pi = \frac{22}{7}$.)

6. A train goes a miles in k hours; how far does it go in t seconds?

7. A point goes a inches in k minutes; how far does it go in an hour?

8. What is the velocity in ft.-sec. units of the end of the minute hand of a clock which is one foot long? ($\pi = \frac{22}{7}$.)

9. What is the velocity in ft.-sec. units of the end of the hour hand of a clock which is 7 inches long? ($\pi = \frac{22}{7}$.)

10. A man bicycles at the rate of 20 miles an hour for the first 2 hours, 15 m. an hour for the next 3 hours, 12 m. an hour for the next 4 hours, and 7 m. an hour during the last hour of his ride: find his average velocity.

11. Compare a velocity of 10 ft. per second with a velocity of 10 in the mile-hour system.

12. A coach travels $80\frac{1}{4}$ miles in 8 hours and 55 minutes; find its average velocity in foot-sec. units.

13. The report of a cannon is heard 48 seconds after it is fired; how far off is it, if the velocity of sound be 1100 ft. per second?

14. How long would a torpedo boat steaming 25 knots take to steam round the earth? [1 knot $= \frac{1}{60}$ of a degree on the earth's surface.]

15. Compare the velocities of the ends of the hour, minute, and second hands of a watch, all of the same length.

16. If u be the measure of a velocity when t seconds and s feet are respectively the units of time and length, find its measure when these units are changed to t' seconds and s' feet.

17. A velocity of 40 miles an hour is four times a velocity of v feet per second: find v.

18. A point moves with a velocity whose measure is 88, and describes 30 miles in $2\frac{1}{2}$ hours: find the unit of length when 4 seconds is the unit of time.

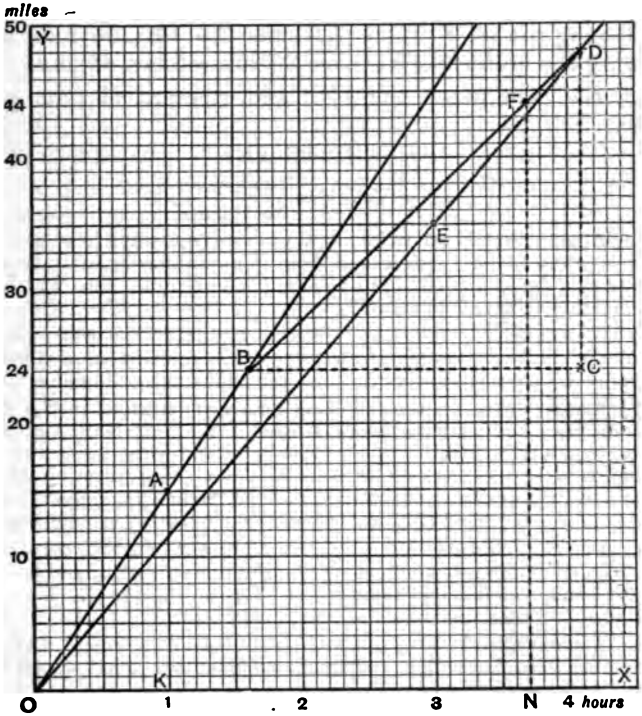
8. *A man starting at noon travels 24 miles at the uniform rate of 15 miles an hour, and does the next 24 miles in $2\frac{1}{2}$ hours. Find graphically, as accurately as you can:—*

(1) *his average velocity in miles per hour during the whole journey,*

(2) *at what time he passes the 44th mile-stone.*

Using paper ruled in inches and tenths of inches (the figure is reduced in printing), take OX as time-axis, and

let one inch represent an hour. Taking OY at right angles to OX as distance-axis, let one inch represent 10 miles.



Along OX take $OK = 1$ hour (1 in.).

Perp. to OX take $KA = 15$ miles (1·5 in.). Join OA.

OA produced is the graph of the first half of the journey.

Take B the pt. where OA cuts the 24 mile line.

Parallel to OX take $BC = 2\frac{1}{2}$ hrs. ($2\frac{1}{2}$ in.).

" " OY " $CD = 24$ miles (2·4 in.). Join BD, nd OD.

BD is the graph of the second half of the journey, and)D is the graph of the man's average velocity.

We see that at E the ordinate $= 35$ miles
and the abscissa $= 3$ hours,

$\therefore$ the average velocity reqd. $= 11{\cdot}7$ m. per hr. approx.

Taking the pt. F where BD cuts the 44 mile line, and drawing the ordinate FN, $ON = 3{\cdot}7$ hrs. $= 3$ hrs. 42 min.

$\therefore$ the man passes the 44th mile-stone at 42 min. past three.

EXAMPLES I *b*.

$$\left[\text{Average Velocity} = \frac{\text{total space}}{\text{total time}}.\right]$$

1. A man travels 55 miles at a uniform rate in 6 hrs. 36 min. Draw a graph of his motion, and from it determine as accurately as you can how far he travels (1) in $2\frac{1}{2}$ hrs., (2) in $5\frac{1}{2}$ hrs. Also find how long he takes to travel 40 miles.

2. A man travels 7 miles in $1\frac{1}{2}$ hours, 18 miles in the next 2 hours, 14 miles in the next 3 hours. Draw graphs of these velocities, and hence determine his average velocity in miles per hour during the whole time.

3. A man travels in the following manner :—9 miles in the first $1\frac{1}{2}$ hours, then a rest for one hour, 11 miles in the next hour, and then half-an-hour's rest, 20 miles in the next hour, and without any further interval for rest does the next 18 miles in 2 hrs. Draw a graph of his motion, and determine to the nearest mile per hour his average velocity during the whole period of time. Check your result by calculation.

4. One man travels uniformly from a place A in a N.E. direction, and a second man starting from B, 40 miles due east of A, and travelling uniformly catches the first man at C, 35 miles from A. How far did the second man travel, and in what direction?

5. A train travels 30 miles in 80 minutes, and a second starting 20 minutes later does the journey in 36 minutes. Find where the second train passes the first.

6. A man travels 6, 9, 12, and 15 miles in each of four consecutive hours. Draw a graph of his motion, and the graph of his average velocity. Determine his average velocity.

7. A is a point 20 yards due east of B. A hare runs from B in a north-easterly direction, and having run 50 yards is caught by a dog which started from A at the same instant. Find graphically the average velocity of the dog if it ran for 4 seconds.

8. A wheel of radius 3 feet makes 50 revolutions a minute ; find the velocity of a point on its rim in feet per second to three significant figures. ($\pi = 3{\cdot}1416$.)

9. A train travels at the rate of 35 miles an hour. Find its velocity in feet per second to three significant digits.

10. Express a velocity of 20 metres per second, in miles per hour. (1 centimetre $= {\cdot}033$ feet.)

CHAPTER II.

ACCELERATION.

9. Def. *The* **acceleration** *of a moving point is the rate of change of its velocity.*

When **uniform, acceleration is measured by** the change of velocity in unit time.

When **variable, acceleration is measured** at any instant by the change of velocity which would take place in unit time if the acceleration remained during that time the same as at the instant under consideration.

At present we shall only deal with uniform accelerations.

10. A point is said to move with **unit acceleration** when its velocity is changed by the unit of velocity in each unit of time.

Thus if ft.-sec. units be used, a point moves with unit acceleration when its velocity is increased by one foot per second during every second.

Or again, if a body has an acceleration 6 ft.-sec. units we mean that in every second of its motion its velocity is increased by 6 ft. per second.

A point is said to have a *negative acceleration* when its velocity is *decreasing*. A negative acceleration is therefore the same as a *retardation*.

11. *A point, starting with velocity u, moves in a straight line subject to a constant acceleration f in its direction of motion: if v be its velocity at time t, to prove that*

$$\underline{\mathbf{v = u + ft.}}$$

By the definition of acceleration, f denotes the change of velocity in each unit of time; therefore ft denotes the change in t units of time. Hence, since the point had an initial velocity, u, its velocity in time t is $u + ft$,

i.e. $$v = u + ft.$$

12. *With the same hypotheses as in the preceding article, if s be the space described in time t; prove that*

$$\mathbf{s = ut + \tfrac{1}{2}ft^2}.$$

Let the time t be divided into n intervals each equal to τ; so that $t = n\tau$.

The velocities of the point at the *beginnings* of these intervals are respectively

$$u,\ u + f\tau,\ u + 2f\tau,\ u + 3f\tau \ldots\ldots \{u + \overline{n-1}\, f\tau\}.$$

Hence if the point moved uniformly during each interval with the velocity which it had at the beginning of that interval, the space described would be:—

$$u\tau + (u + f\tau)\,\tau + (u + 2f\tau)\,\tau + \ldots\ldots + \{u + \overline{n-1}\, f\tau\}\,\tau$$

$$= nu\tau + f\tau^2\,\{1 + 2 + \ldots\ldots + \overline{n-1}\}$$

$$= nu\tau + f\tau^2\,\frac{n\,\overline{n-1}}{2}$$

$$= ut + \tfrac{1}{2}ft^2\left(1 - \frac{1}{n}\right) \text{ since } \tau = \frac{t}{n}. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again the velocities of the point at the *ends* of these intervals are respectively

$$u + f\tau,\ u + 2f\tau,\ \ldots\ldots\ u + nf\tau.$$

Hence if the point moved uniformly during each interval with the velocity which it had at the end of that interval, the space described would be:—

$$(u + f\tau)\,\tau + (u + 2f\tau)\,\tau + \ldots\ldots + (u + nf\tau)\,\tau$$

$$= nu\tau + f\tau^2\,(1 + 2 + \ldots\ldots + n)$$

$$= nu\tau + f\tau^2\,\frac{n\,.\,\overline{n+1}}{2}, \quad = ut + \tfrac{1}{2}ft^2\left(1 + \frac{1}{n}\right) \ldots\ldots\ldots\ldots\ldots(2).$$

Now the actual space described by the point lies *between* the values found in (1) and (2), i.e. between

$$ut + \tfrac{1}{2}ft^2\left(1 - \frac{1}{n}\right) \text{ and } ut + \tfrac{1}{2}ft^2\left(1 + \frac{1}{n}\right),$$

therefore making n, the number of intervals, infinitely large, but keeping t the same, $\frac{1}{n}$ becomes zero and the values in

(1) and (2) both approximate to, and ultimately coincide with

$$ut + \tfrac{1}{2}ft^2,$$

i.e.

$$\mathbf{s = ut + \tfrac{1}{2}ft^2}.$$

13. *With the same hypotheses to prove the formula*

$$\mathbf{v^2 = u^2 + 2fs}.$$

$$v^2 = (u + ft)^2 \qquad \text{(Art. 11)}$$
$$= u^2 + 2f(ut + \tfrac{1}{2}ft^2)$$
$$= u^2 + 2fs. \qquad \text{(Art. 12.)}$$

14. *To find the space described in a particular second.*

[The beginner must be careful to distinguish between the space described in t seconds, and the space described in the t^{th} second of the motion of a particle.]

The space described in the t^{th} second

= the space described in t seconds − space described in $(t-1)$ seconds,

$$= [ut + \tfrac{1}{2}ft^2] - [u(t-1) + \tfrac{1}{2}f(t-1)^2]$$
$$= u + \tfrac{1}{2}f[t^2 - (t-1)^2]$$
$$= u + \tfrac{1}{2}f(2t-1).$$

15. When the point starts from rest $u = 0$ and the formulae in Arts. 11, 12, 13 become respectively

$$v = ft.$$
$$s = \tfrac{1}{2}ft^2.$$
$$v^2 = 2fs.$$

16. Ex. i. *A point moving with a velocity of* 128 *ft. per sec. is brought to rest in* 16 *secs. by a uniform retardation: find this retardation.*

Let f be the retardation in ft.-sec. units.

$$\therefore 0 = 128 - 16f, \qquad [v = u + ft]$$
$$\therefore f = 8 \text{ ft.-sec. units.}$$

Ex. ii. *A particle is moving with uniform acceleration; in the* 8*th second of its motion it moves through* 47 *feet, and in the* 11*th second it moves through* 65 *feet; find its initial velocity and its acceleration.*

Let u be its initial velocity and f its acceleration. Then 47 = space described in 8 secs. − the space described in 7 secs.

$$= 8u + \tfrac{1}{2}f(8)^2 - [7u + \tfrac{1}{2}f(7)^2] \qquad (s = ut + \tfrac{1}{2}ft^2)$$
$$= u + \tfrac{1}{2}f(8^2 - 7^2) \qquad [8^2 - 7^2 = (8-7)(8+7)]$$

or $$47 = u + \tfrac{15}{2}f \text{ ....} (1).$$

Also $65 = 11u + \frac{1}{2}f(11)^2 - [10u + \frac{1}{2}f(10)^2]$

$= u + \frac{1}{2}f(11^2 - 10^2)$

or $65 = u + \frac{21}{2}f$ ....(2).

Therefore subtracting (1) and (2) $3f = 18$, $f = 6$ ft.-sec. units.

Substituting in either (1) or (2) $u = 2$ ft. per sec.

17. Change of units. *If f be the measure of an acceleration when t seconds and s feet are respectively the units of time and length, find its measure when these units are changed to t' seconds and s' feet.*

Here, acceleration f means that the velocity is increased by f times s feet per t seconds during every t seconds.

i.e. by $\frac{fs}{t}$ feet per second during every t seconds.

i.e. by $\frac{fs}{t^2}$ feet per second during every second.

i.e. by $\frac{fs}{s't^2}$ times s' feet per second during every second.

i.e. by $\frac{f \cdot s \cdot t'^2}{s' \cdot t^2}$ times s' feet per t' seconds during every t' seconds.

$\therefore \frac{f \cdot s \cdot t'^2}{s' \cdot t^2}$ is the measure of the acceleration when t' secs. and s' feet are respectively the units of time and space.

Example. Express an acceleration of 32 ft.-sec. units in mile-hour units.

An acceleration of 32 ft.-sec. units

= an increase of velocity of 32 ft. per sec. during every second

= .... $\frac{32}{1760 \times 3}$ miles ....

= .... $\frac{32 \times 60^2}{1760 \times 3}$ " per hour ....

= .... $\frac{32 \times 60^2 \times 60^2}{1760 \times 3}$ miles per hour during every hour.

$\therefore$ an acceleration of 32 ft.-sec. units

= an acceleration $\frac{32 \times 60^2 \times 60^2}{1760 \times 3}$ mile-hour units,

= .... $78545\frac{5}{11}$ mile-hour units.

In connection with any *change of units* it will be a very great help if the beginner grasps thoroughly the fact that in all cases:—

The measure of any quantity varies inversely as the unit in which it is measured.

Thus $\qquad 4 \text{ miles} = 8 \text{ half-miles},$

i.e. if we halve the unit (use a half-mile as unit instead of a mile) the measure is doubled, it is 8 instead of 4.

In the same way, $4 \text{ yards} = 12 \text{ feet},$

i.e. if we use $\frac{1}{3}$ yard as unit instead of 1 yard, the measure is multiplied by 3; it is 12 instead of 4. This will be seen to hold for all units.

EXAMPLES. II *a*.

[Note that a velocity of 60 miles an hour $= \dfrac{60 \times 1760 \times 3}{60 \times 60}$ ft. per sec.
$= 88$ ft. per sec.]

1. A body starting from rest moves with an acceleration of 2 ft.-sec. units; find its velocity in 4 secs., and the space described in that time.

2. A point, starting with a velocity of 8 ft. per sec., has a velocity of 2 ft. per sec. at the end of 2 seconds: find its acceleration and the space described in this time.

3. A body moving with an acceleration of 8 ft.-sec. units has a velocity of 32 ft. per sec. when it has described 16 ft. Find its initial velocity.

4. What is the uniform acceleration of a body which, starting from rest, describes 500 feet in 5 seconds?

5. How long does a body, starting with a velocity of 8 ft. per sec. and moving with an acceleration of 5 ft.-sec. units, take to acquire a velocity of 60 miles an hour?

6. In what time does a body, starting with a velocity of 5 ft. per sec. and moving with an acceleration of 8 ft.-sec. units, describe a space of 150 yards? Has the negative answer any meaning?

7. A point starts with a velocity of 10 cms. per sec. and moves with a retardation of 5 centimetre-sec. units. When and where will it come to rest?

8. A particle moving with uniform acceleration describes 260 feet in the seventh second of its motion. Find its acceleration if it started from rest.

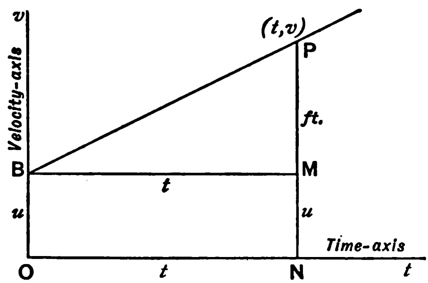
18. *Geometrical representation of $v = u + ft$.*

Take Ot as time-axis, and Ov at right angles to Ot a velocity-axis. Plot the point (t, v), P in the diagram draw the ordinate PN.

In Ov take $OB = u$, the initial velocity, and draw parallel to Ot to meet PN at M.

Then $\dfrac{PM}{BM} = \dfrac{PN - MN}{BM} = \dfrac{v - u}{t} = \dfrac{ft}{t} = f$, which is con

$\therefore$ the locus of the point (t, v) is a straight line th the point B on the velocity-axis.



N.B. The slope of $BP = \tan PBM = f$, the accelera

19. *Graphical proof of the formula $s = ut + \frac{1}{2}ft^2$.*

Take Ot as time-axis, and Ov at right angles to velocity-axis.

Let ON represent any time t, and divide it into n portions OA_1, A_1A_2, A_2A_3.... Let OP represent the velocity, and draw the ordinates OP, A_1a_1, A_2a_2 represent the velocities of the particle at the times OA_2 ....

The points P, a_1, a_2,... lie on the straight line PQ, is the graph of $v = u + ft$. (See preceding article.)

Complete the rectangles as shown in the diagram.

On the supposition that the point moves uniformly (

128 feet before coming to rest. If the retardation continues to act, what happens to the point after this?

22. A point, subject to a uniform acceleration, passes over 260 ft. in a certain interval of 2 seconds, and comes to rest after 4 seconds more; how much further did it move?

23. The measure of an acceleration in ft.-sec. units is 2; find its measure when a minute is the unit of time.

24. Express an acceleration of 3 mile-minute units in foot-sec. units.

25. An acceleration of 64 ft.-sec. units is expressed by 4 in 'x ft.-2 sec.' units: find x.

26. Compare an acceleration of 8 ft.-sec. units with an acceleration 8 in a system of '3 ft.-4 sec.' units.

27. If the measure of an acceleration is f, what change in this measure is produced by taking double the unit of time as the new unit?

28. A particle starts from a point O with a uniform velocity of 4 ft. per sec., and after 2 secs. another particle leaves O in the same direction with a velocity of 5 ft. per sec. and subject to an acceleration of 3 ft.-sec. units. Find when and where it will overtake the first particle.

29. A body moving with a constant acceleration is observed to move over spaces a, b during the m^{th} and n^{th} seconds respectively from the commencement of motion; find the acceleration and initial velocity.

30. In a certain interval of 10 secs. a point passes over 220 ft., in the next interval of 5 secs. it passes over 330 ft.; assuming that the point is moving with uniform acceleration, find its velocity at the beginning of each of the two intervals.

31. In travelling a mile the velocity of a body increases from 200 to 240 ft. a second. What is the acceleration? What is the velocity half way? and also at half time?

32. Two particles start simultaneously from the same point to traverse a closed circular path, 144 feet long, in opposite directions: the one moves with the uniform velocity of 7 ft. per sec. and the other, whose initial velocity is zero, has a uniform acceleration of 2 ft.-sec. units. When and where do they meet?

33. A particle leaves a point A with a certain velocity and with an acceleration 3 ft.-sec. units in the opposite direction. At the end of 2 secs. this acceleration is suddenly increased to 12 ft.-sec. units, and after another 2 secs. the particle is again at A. Find for how long and how far the particle travels forwards, and the velocity of its departure from and return to A.

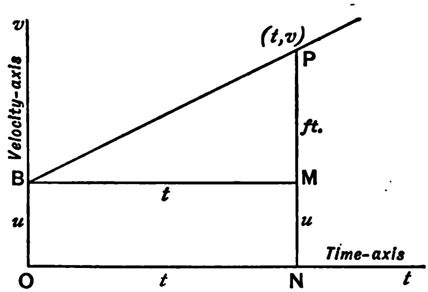
18. *Geometrical representation of $v = u + ft$.*

Take Ot as time-axis, and Ov at right angles to Ot as the velocity-axis. Plot the point (t, v), P in the diagram, and draw the ordinate PN.

In Ov take $OB = u$, the initial velocity, and draw BM parallel to Ot to meet PN at M.

Then $\dfrac{PM}{BM} = \dfrac{PN - MN}{BM} = \dfrac{v - u}{t} = \dfrac{ft}{t} = f$, which is constant.

$\therefore$ the locus of the point (t, v) is a straight line through the point B on the velocity-axis.



N.B. The slope of $BP = \tan PBM = f$, the acceleration.

19. *Graphical proof of the formula $s = ut + \frac{1}{2}ft^2$.*

Take Ot as time-axis, and Ov at right angles to Ot as velocity-axis.

Let ON represent any time t, and divide it into n equal portions OA_1, A_1A_2, A_2A_3.... Let OP represent the initial velocity, and draw the ordinates OP, A_1a_1, A_2a_2 ... to represent the velocities of the particle at the times OA_1, OA_2 ....

The points P, a_1, a_2,... lie on the straight line PQ, which is the graph of $v = u + ft$. (See preceding article.)

Complete the rectangles as shown in the diagram.

On the supposition that the point moves uniformly during

each interval with the velocity which it had at the beginning of that interval,

the rect. PA_1 represents the space described in the first interval,

the rect. a_1A_2 represents the space described in the second interval,

the rect. a_2A_3 represents the space described in the third interval,

and so on.

$\therefore$ on this supposition, the space described by the particle in time t is represented by the sum of the rects. PA_1, $a_1A_2, \ldots.$

In the same way, on the supposition that the particle moves uniformly during each interval with the velocity which it had *at the end of that interval*:

the space described will be represented by the sum of the larger rectangles Oa_1, $A_1a_2, \ldots.$

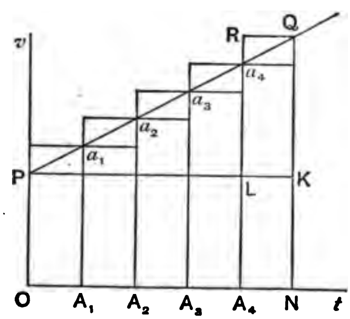
Now, without altering the total time t, let the number of intervals be increased indefinitely, so that the length of each interval is decreased indefinitely. Then we see that each of the two *sums* under the above hypotheses approximates to, and ultimately coincides with the figure $PONQ$.

$\therefore$ $PONQ$ represents the actual space described.

Hence $s = PONK + \triangle PQK$

$$= OP \,.\, ON + \tfrac{1}{2}PK \,.\, QK$$

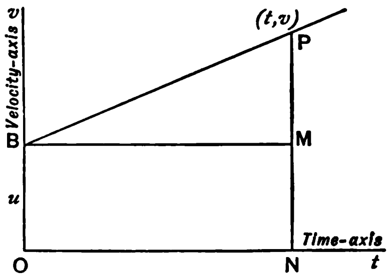
$$= ut + \tfrac{1}{2}ft^2 \quad (\text{for } QK = v - u = ft).$$



N.B. The difference of the sums of the two series of rectangles is equal to the rect. $RQKL$, whose area indefinitely decreases as the interval A_4N becomes indefinitely small.

The proof when $u=0$ will be similar to the above, the straight line PQ passing through the origin in that case.

20. *Geometrical proof of the formula* $v^2=u^2+2fs$.



With the same diagram and construction as in Art. 19, the area $BONP$ represents s, and we have to find the relation between s, $u\,(OB)$ and $v\,(PN)$.

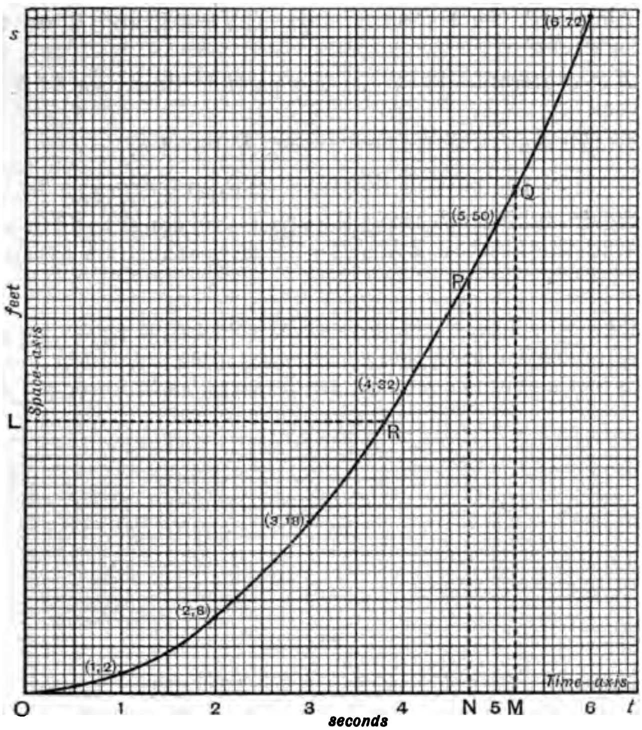
$$s = \text{area } BONP = \tfrac{1}{2}(PN+OB)\,ON$$
$$= \tfrac{1}{2}(v+u)\,BM = \tfrac{1}{2}(v+u)\,PM \cot PBM$$
$$= \tfrac{1}{2}(v+u)\frac{(v-u)}{f}, \text{ for } f=\tan PBM \text{ (Art. 18)}$$
$$= \frac{v^2-u^2}{2f},$$
$$\therefore\ v^2=u^2+2fs.$$

21. *A particle moves from rest under an acceleration of* 4 *ft.-sec. units. Draw a graph of its space-time equation* $[s=\tfrac{1}{2}ft^2]$. *From the figure write down:*

(1) the space described in 4·7 secs.

(2) .... 5·2 ...

(3) the time taken to travel 29 feet.



$$f = 4. \quad \therefore \; s = \tfrac{1}{2} \times 4t^2 = 2t^2.$$

$\therefore$ When	$t=1$	2	3	4	5	6
	$s=2t^2=2$	8	18	32	50	72

Using one inch to represent a sec. along the time-axis, and one-tenth of an inch to represent one foot along the space-axis, we plot the points

$$(1, 2),\ (2, 8),\ (3, 18),\ (4, 32),\ (5, 50),\ (6, 72)$$

as in the figure.

[The figure is considerably reduced in printing.]

Along Ot the time-axis, take $ON = 4{\cdot}7$, and draw the ordinate NP.

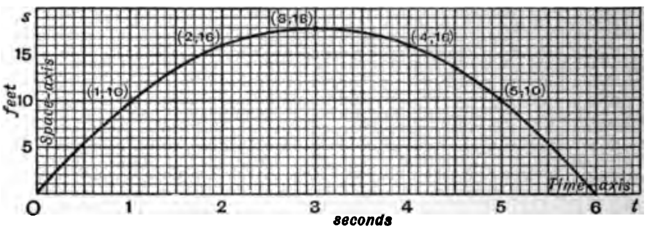
NP represents 44 ft. $\therefore$ the particle describes 44 ft. in 4·7 secs.

In the same way $OM = 5{\cdot}2$ secs., $MQ = 54$ ft.

$\therefore$ the particle describes 54 ft. in 5·2 secs.

Again, taking OL along the space-axis equal to 29 ft. and drawing the abscissa LR, $LR = 3{\cdot}8$ secs. = the time taken to travel 29 ft.

22. *A particle with an initial velocity of* 12 *ft. per sec. moves under a retardation of* 4 *ft.-sec. units. Draw a graph of its motion for six seconds, using the formula* $s = ut + \frac{1}{2}ft^2$.



Here $f = -4$, $u = 12$. $\therefore$ $s = 12t - 2t^2$.

When

$t =$	0	1	2	3	4	5	6
$12t =$	0	12	24	36	48	60	72
$2t^2 =$	0	2	8	18	32	50	72
$s =$	0	10	16	18	16	10	0

Using one inch to represent a sec. along the time-axis, and one-tenth of an inch to represent one foot along the space-axis, we plot the points

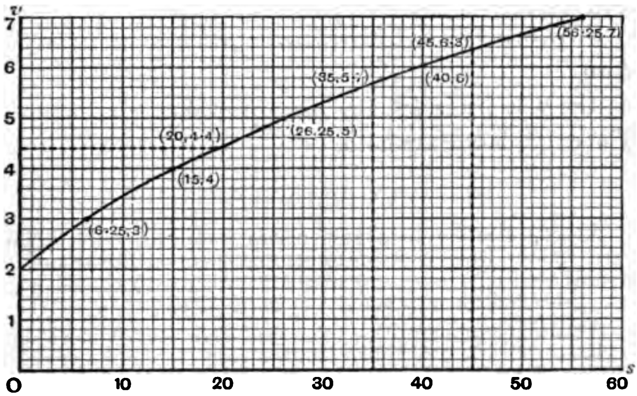
(1, 10), (2, 16), (3, 18), (4, 16), (5, 10), (6, 0).

[The figure is reduced in printing.]

From the diagram we see that in the first three seconds, he particle is moving in the direction of its initial velocity; *fter* the first three secs. it moves in the opposite direction *oming to rest at the starting-point in* 6 *secs.*

As in the preceding example, we might write down the pace described in any given time, or the time required to escribe any given space.

23. *Geometrical representation of* $v^2 = u^2 + 2fs$.



We will take the case when $u = 2$, and $f = \cdot 4$, so that $^2 = 4 + \cdot 8s$, $\cdot 8s = v^2 - 4$, $s = \dfrac{v^2 - 4}{\cdot 8}$, and we see that v cannot e less than 2.

When $v = 2$	3	4	5	6	7...
$v^2 - 4 = 0$	5	12	21	32	45...
$s = \dfrac{v^2 - 4}{\cdot 8} = 0$	6·25	15	26·25	40	56·25...

Take the space-axis across the page, using one-tenth of n inch to represent unity. (The figure is reduced in rinting.)

Take the velocity-axis up the page, using half-an-inch to epresent the unity of velocity.

Plot as accurately as possible the points

$(0, 2)$, $(3, 6{\cdot}25)$, $(4, 15)$, $(5, 26{\cdot}25)$, $(6, 40)$, $(7, 56{\cdot}25)$,

and join them by an even curve.

This curve is the graph of the equation $v^2 = 4 + {\cdot}8s$.

From the figure we can read off the velocity of the particle when it has described a given space.

Thus

when a space 35 has been passed over, the velocity $= 5{\cdot}7$
.... 45 .... $= 6{\cdot}3$.

Also

when the velocity is $4{\cdot}4$, the space passed over $= 20$ approx.

The curve is a **Parabola.**

EXAMPLES II *b*.

1. A body starting from rest moves under an acceleration of ·5 ft.-sec. units. Draw a graph to determine its velocity at any time, and from it read off the velocity of the body after 17 secs. and after 45 secs. Also find when its velocity is 28 ft. per sec.

2. A particle starting with a velocity of 30 ft. per sec. moves under a retardation of ·6 ft.-sec. units. Draw a graph to find its velocity at any time, and hence determine, as accurately as you can, its velocity in 37 secs. and when it comes to rest.

3. A particle moving under a uniform acceleration has its velocity increased from 10 ft. per sec. to 41 ft. per sec. in 60 secs. Draw a graph to determine its velocity at any time, and read off, as accurately as you can, its velocity in 35 secs., and the time when its velocity is 28 ft. per sec.

4. A body starting from rest moves under an acceleration of ·2 ft.-sec. units. Draw a graph from which you can determine its velocity when any given space (up to 60 feet) has been described. Write down, as accurately as you can, the velocity of the body when it has passed over (1) 10 ft., (2) 30 ft.

5. A body starting with a velocity of 8 ft. per sec. is subject to a retardation of ·5 ft.-sec. units. Draw a graph which will give its velocity when a given space has been described, and hence determine the velocity of the body when it has passed over (1) 20 ft., (2) 40 ft.

6. A particle starting from rest moves under an acceleration of ·6 ft.-sec. units. Draw a graph to determine the space passed over in any given time, and read off the space described in 3·4 secs., and in 4·8 secs.; also the time taken to travel 6 feet.

7. A particle, starting with a velocity of 3 ft. per sec., moves with an acceleration of 4 ft.-sec. units. Plot a graph to determine the space described in any time, and read off the space described in 2·2 secs. and in 3·5 secs. How long does the particle take to describe 25 ft.?

8. After t seconds it is found that a particle has passed over a space s feet reckoned from some starting point, and it is known that $s=10+8t-2t^2$. Plot the graph of this equation for the first 6 seconds of the motion. When, during this time, is the particle furthest from its starting-point, and where is it in 5 seconds?

9. The velocity (v ft. per sec.) of a body after t seconds of motion is given by the table below.

t	1	2	3	4	5	...
v	18	14	10	6	2	...

Plot the corresponding values of t and v. What do you conclude as to the acceleration of the body? What was its initial velocity, and when does it come to rest?

24. *If a particle describes space s in time t, and space $s+\Delta s$ in time $t+\Delta t$, to prove that its*

$$\textbf{velocity at time } t = \textbf{the limiting value of } \frac{\Delta s}{\Delta t},$$

when Δs and Δt are indefinitely small.

Let v be the velocity of the particle at time t, v' its velocity at time $t+\Delta t$.

Then Δs, the space described in time Δt, lies between $v\Delta t$ and $v'\Delta t$. $(s=ut.)$

$\therefore \dfrac{\Delta s}{\Delta t}$ lies between v and v'.

But as Δs and Δt grow smaller, v' approaches to v.

$\therefore$ the limiting value of $\dfrac{\Delta s}{\Delta t}=v$, the velocity at time t.

Q. E. D.

If v is the velocity of a particle at time t, $v+\Delta v$ its velocity at time $t+\Delta t$, to prove that its

$$\textbf{acceleration at time } t = \textbf{the limiting value of } \frac{\Delta v}{\Delta t},$$

when Δv, and Δt are indefinitely small.

If f is the acceleration of the particle at time t, and f' its acceleration at time $t+\Delta t$,

Δv, the change of velocity in time Δt, lies between

$$f\Delta t \text{ and } f'\Delta t \quad (v-u=ft),$$

$$\therefore \frac{\Delta v}{\Delta t} \text{ lies between } f \text{ and } f'.$$

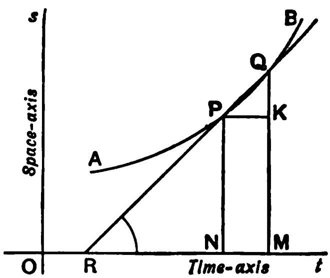
But as Δv and Δt grow smaller, f' approaches to f,

$\therefore$ the limiting value of $\dfrac{\Delta v}{\Delta t}=f$, the acceleration at time t.

Q. E. D.

25. *If the time-space curve of a moving point is drawn, the velocity of the point at any time is equal to the* **slope** *of the curve at the point on it corresponding to the given time.*

Let APB be part of the time-space curve of a moving point, Ot being the time-axis, and Ov the velocity-axis.



Take two points on the curve, P and Q, near to one another, and let Δt represent the time PK, and Δs the space KQ in the diagram.

Then, as in Art. 24, the velocity at P will be equal to the ultimate value of $\dfrac{\Delta s}{\Delta t}$, when we decrease Δs and Δt indefinitely.

Hence the velocity at P

$=$ the ultimate value of $\dfrac{\Delta s}{\Delta t}$, when Q coincides with P,

$=$ the ultimate value of $\tan QPK$, when Q coincides with P,

$=$ the **slope** of the curve at P.

N.B. If the moving point has a uniform acceleration, the time-space curve is a parabola, whose axis is parallel to the space-axis.

26. *The equation connecting s, the space passed over, and t, the time of motion of a moving particle is* $s = 84 + 18t - 2t^2$. *Find the velocity of the particle at any time t.*

If $s + \Delta s$ is the space passed over in time $t + \Delta t$,

$$s + \Delta s = 84 + 18(t + \Delta t) - 2(t + \Delta t)^2.$$

Also $$s = 84 + 18t - 2t^2,$$

$\therefore$ subtracting, $$\Delta s = 18\Delta t - 4t\Delta t - 2(\Delta t)^2,$$

and $$\frac{\Delta s}{\Delta t} = 18 - 4t - 2\Delta t.$$

Hence, diminishing Δs and Δt indefinitely,

$$\text{the velocity at time } t = \text{the limiting value of } \frac{\Delta s}{\Delta t}$$
$$= 18 - 4t.$$

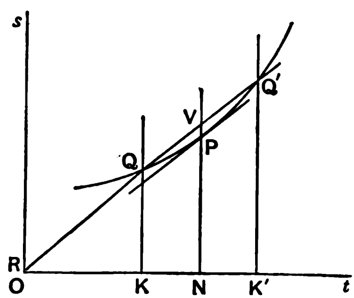
N.B. In each unit of time, the velocity of the particle is decreased by 4,

$\therefore$ it is subject to a retardation whose measure is 4.

This might also be seen by using the method of the second part of Art. 24. It is left as an exercise.

27. *From the graph of* $s = ut + \frac{1}{2}ft^2$, *to find the velocity at any time t.*

We know the graph to be a parabola whose axis is parallel to the space-axis Os.



Let P be the point on the curve corresponding to time t.

Draw the ordinate PN, and in Ot make $NK = NK'$. Also draw ordinates KQ, $K'Q'$ to meet the curve in Q, Q'. Join QQ', cutting PN at V.

$KN = K'N$, $\therefore QV = Q'V$.

$\therefore PV$ is a diameter of the parabola bisecting the chord QQ'.

$\therefore$ the tangent at P is parallel to QQ'.

Hence if QQ' meets the time-axis at R

$$\tan QRt = \text{the velocity at time } t, \text{ by Art. 25.}$$

The above method is very easily applied when squared paper is used. For example, to find the velocity after three seconds, we only have to join the 2 sec. and 4 sec. points on the curve, and read off the tangent of the angle this line makes with the time-axis.

EXAMPLES II *a*

1. The space-time equation (ft.-secs.) of a particle moving in a straight line is $s=50t-4t^2$. Plot the graph of the equation up to $t=4$.

Deduce the velocity of the particle after 1 second, after 2 seconds, and after 3 seconds. Verify your results by another method.

Also prove that the particle is subject to a uniform acceleration, and find its value.

2. The space-time equation (ft.-secs.) of a particle moving in a straight line is $s=5+2t^2$. Plot the graph of this equation up to $t=5$. Deduce the velocity of the particle after 2 secs., after $2\frac{1}{2}$ secs., and after 4 secs. Verify by another method. Prove that the particle is subject to a uniform acceleration and find its value.

3. The position of a point P on a straight line is given by $x=t^2-2t+3$, where t denotes the number of seconds after a fixed instant of time, and x the distance in feet of P from a fixed point O on the line. What are the positions of P at the fixed instant, and at intervals of 1 and 3 secs. after the fixed instant?

What space does P traverse between time 3 secs. and time 6 secs., and between times 3 secs. and 3·001 secs., and what is the average speed for each of these intervals? What is the actual speed at time 3 secs?

4. At the end of a time t secs. it is observed that a body has passed over a distance s feet reckoned from some starting point. If it is known that $s=25+150t-5t^2$, what is the velocity at the time t?

If corresponding values of s and t are plotted on squared paper, what indicates the velocity and why?

5. The space-time equation of a moving particle is

$$x = at + bt^2.$$

Find the velocity of the particle at time t, and prove that it is subject to an acceleration which is constant and measured by $2b$.

6. A particle moves in a straight line and its distance, x, from a given point at time t, is given by the equation $x = 2t^3$.

Find its velocity at time t, and prove that it is subject to an acceleration which varies as t.

CHAPTER III.

VERTICAL MOTION UNDER GRAVITY.

28. IT can be proved by experiment that if a number of bodies, of different sizes, shapes, and weights, be allowed to fall from rest in a vacuum [such as the exhausted receiver of an air-pump], starting simultaneously from different points in the same horizontal plane, they will move with uniformly increasing velocities and will describe equal vertical spaces in the same time. They must all therefore be moving with the same acceleration.

This acceleration is called the **acceleration due to gravity** or the **acceleration of gravity.** It is found to be always the same at the same place on the earth's surface, but it varies slightly at different places.

We usually employ the letter "g" to denote this acceleration: its value at Greenwich has been found by experiment to be 32·19... ft.-sec. units.

N.B. In all numerical examples, unless it is otherwise stated, the value of "g" in vacuo may be taken to be 32 ft.-sec. units or 981 centimetre-sec. units.

29. As a result of the preceding, we see that if a body be projected vertically *upwards* with velocity u, and move freely under the action of gravity, the three formulae of Arts. 11, 12, 13 become respectively

$$v = u - gt, \qquad (v = u + ft)$$
$$s = ut - \tfrac{1}{2}gt^2, \qquad (s = ut + \tfrac{1}{2}ft^2)$$
$$v^2 = u^2 - 2gs. \qquad (v^2 = u^2 + 2fs)$$

30. If the body be projected vertically *downwards*, the acceleration will be in the same direction as the initial velocity and therefore of the same sign: hence in this case the same formulae respectively become

$$v = u + gt,$$
$$s = ut + \tfrac{1}{2}gt^2,$$
$$v^2 = u^2 + 2gs.$$

31. **Ex. i.** *A body is let fall from the top of a tower, and at the same instant another is projected upwards from the foot of the tower with a velocity of 128 feet per second. They meet half way up; find the height of the tower.*

Let h feet be the height of the tower, and let the bodies meet in t seconds.

Considering the falling body $\frac{h}{2} = \frac{1}{2}gt^2$ ....(i), $(s = ut + \frac{1}{2}ft^2)$

" rising " $\frac{h}{2} = 128t - \frac{1}{2}gt^2$ ......(ii). (" " ")

Adding these equations $h = 128t$.

Substituting for h in equation (i),

$$64t = 16t^2,$$
$$\therefore t = 4 \text{ secs. } (t = 0 \text{ is inadmissible});$$

$\therefore$ from equation (i)

$$h = gt^2$$
$$= 32 \times 16$$
$$= 512 \text{ feet}.$$

Ex. ii. *From a balloon ascending with a velocity of 16 feet per second a ball is let fall, and reaches the ground in 16 seconds; how high was the balloon when the ball left it?*

Let h feet be the height required.

The ball leaves the balloon with a velocity of 16 feet per second upwards,

$\therefore$ taking the upward direction as positive

$$-h = +16 \times 16 - \tfrac{1}{2}g(16)^2 \qquad (s = ut + \tfrac{1}{2}ft^2)$$
$$= +16 \times 16 - 16 \times (16)^2$$
$$= -16 \times 16\,(16 - 1)$$
$$= -3840 \text{ feet}.$$

The balloon was therefore at a height of 3840 feet.

[The negative value of h shews us that, since we took the upward direction as positive, the body is 4896 feet *below* its point of projection in 18 seconds.]

Ex. iii. *Given $g=32{\cdot}2$ ft.-sec. units, find its value in centimetre-sec. units. (A metre $=39{\cdot}37$ inches approx.)*

Acceleration g

= an increase of velocity of 32·2 ft. per sec. during every sec.

= .... $32{\cdot}2\times 12$ in. ....

= .... $\dfrac{32{\cdot}2\times 12}{39{\cdot}37}$ metres ....

= .... $\dfrac{3220\times 1200}{3937}$ centimetres ....

$$\therefore g=\frac{3220\times 1200}{3937}=981 \text{ centimetre-sec. units approx.}$$

N.B. When a ball is thrown vertically upwards:—

(1) its velocity at its highest point is zero,

(2) when it returns to the hand again, the total space described is zero, $(h-h)$.

EXAMPLES. III.

[*Questions* 28—32 *might with advantage be done immediately after Question* 5.]

1. A stone is let fall from a height of 144 feet; in how many seconds will it reach the ground?

2. A stone is projected vertically upwards with a velocity of 256 feet per sec.; when will it come to rest, and how high will it go?

3. A body is thrown vertically upwards with a velocity of 128 feet per second. After what times will its velocity be 96 ft. per sec. and at what height will it then be?

4. A body is thrown vertically downwards from the top of a tower with a velocity of 16 ft. per sec. and reaches the ground in 3 secs.; find the height of the tower.

5. The greatest height attained by a body when projected vertically upwards is 169 feet; with what velocity was it projected?

6. A ball is thrown vertically upwards with a certain velocity; prove that the time of its ascent is equal to that of its descent to the same point.

7. A ball is thrown vertically upwards with velocity v; prove that it returns to the hand with the same velocity.

8. A falling body is observed to pass through 304 feet in the last second of its motion; find the height from which it fell.

9. Given that a body falls through $241\frac{1}{2}$ feet in its 8th second, find the value of g in ft.-sec. units.

10. A body projected vertically downwards describes 720 feet in t secs., and 2240 feet in $2t$ secs.; find t, and the velocity of projection.

11. A particle passes a certain point moving upwards with a velocity of 43·6 metres per second; how long after this will it be moving downwards at the same speed?

12. A ball thrown vertically upwards returns to the hand in 6 seconds; how high does it rise, and when will it be half-way up?

13. A ball is projected vertically upwards from the top of a tower with a velocity of 64 feet per second, and reaches the foot of the tower in 6 seconds; find the height of the tower.

14. A body after falling for some time under the action of gravity is observed to pass through 768 feet in 4 seconds; how far will it fall in the next two seconds?

15. A body after falling for 3 seconds is overtaken by another let fall 640 feet above it: how long has the latter been falling?

16. A body is projected vertically upwards with a velocity of 160 feet per sec.: after what time will it have attained a height of 256 feet. Explain the double result.

17. A body is projected vertically upwards with a certain velocity, and it is found that when in its ascent it is at a point 960 ft. from the ground it takes 4 secs. to return to the same point again; find the velocity of projection and the whole height ascended.

18. A particle starting vertically downwards with a velocity of 100 ft. per sec. acquires a velocity of 90 miles per hour: find the space described.

19. A body is projected vertically upwards with a velocity of 16 feet per second: find its position in 4 seconds.

20. A stone P is thrown vertically upwards with a velocity of 78 ft. per sec. and after 3 seconds another stone Q is let fall from the same point. Prove that P will overtake Q after 5 seconds more.

21. A stone falls freely for 3 secs., when it passes through a sheet of glass and loses half its velocity, and then reaches the ground in 2 seconds: find the height of the glass.

22. A tower is 288 feet high; one body is dropped from the top of the tower and at the same instant another is projected vertically upwards from the bottom, and they meet half-way up; find the initial velocity of the projected body, and its velocity when it meets the descending body.

23. A stone is projected vertically upwards with a velocity sufficient to carry it to height h; find its velocity when it is half-way up.

24. A body is let fall, and 3 secs. afterwards another is projected vertically upwards from the same point with a velocity of 64 ft. per second. Find where it overtakes the first body.

25. A body projected vertically upwards rises through h feet in t seconds: for how much longer will it continue to rise?

26. A body projected vertically upwards passes a point h ft. from the ground with a velocity v feet per sec.; after what time will it reach the ground again?

27. If the measure of gravity acceleration be 1, what is necessary as to the units of time and space?

28. A ball is thrown vertically upwards with a velocity of 128 ft. per second. After what time is it moving with a velocity of 64 ft. per sec. downwards?

[Use the formula $v=u+ft$, remembering that if 128 is its initial velocity, -64 is the velocity at the required time.]

29. A ball is thrown vertically upwards from the top of a tower with a velocity of 58 ft. per sec. When will it be 24 ft. below its point of projection?

[Use the formula $s=ut+\frac{1}{2}ft^2$, remembering that $s=-24$ at the reqd. time, if we take the upward direction as positive.]

30. A ball is thrown vertically upwards with a velocity of 88 ft. per sec. What time elapses before it returns to the hand again?

31. From a balloon rising vertically with a velocity of 20 ft. per sec. and at a height of 456 feet, a stone is dropped. How long does it take to reach the ground?

32. A stone is dropped from rest from the top of a tower 160 ft. high. Neglecting the frictional resistance of the air, find how long it takes to reach the ground, and calculate its velocity just before it reaches the ground. Plot a curve roughly showing the velocity of the stone at various heights.

CHAPTER IV.

LAWS OF MOTION.

32. Matter. *Matter* can only be considered as a primary conception of the mind. Like time and space, it does not admit of any satisfactory definition.

A *particle* is a portion of matter whose dimensions are so small that it may be treated as a physical point.

Mass. The *mass* of a body or particle is the quantity of matter in it.

The standard British unit of mass is a piece of platinum kept at the Exchequer Office, and is called a *pound.*

Force. *Force* is that which changes or tends to change the state of rest or uniform motion of a body.

The unit of force is that force which acting on the unit of mass produces the unit of acceleration.

Weight. The *weight* of a body is the force with which the earth attracts the body.

It is extremely important that the beginner should at once grasp the fact that *weight* and *mass* are not synonymous terms. This is easily seen when we remember that the *mass* of a body is the same wherever the body may be situated, whereas the *weight* of a body varies with its position.

According to the universal Law of Gravitation, the attraction of the earth upon a body outside it varies inversely as the square of the distance of the body from the centre of the earth; since therefore the earth is not a perfect sphere, it is found that the weight of a body is different at different

places. Thus if a body be weighed by means of a spring balance, first at a point on the equator, and then at some point north or south of the equator, its weight in this latter position will be found to be greater than at the equator—it being nearer to the centre of the earth.

Or again, if we imagine a body removed to an infinite distance from any other body its mass (i.e. the quantity of matter in it) would remain unaltered but its weight would be nil.

Momentum. The *momentum*, or quantity of motion of a particle, is the product of its mass and its velocity. It is denoted by mv where m is the mass and v the velocity of the particle.

The unit of momentum is therefore the momentum of the unit of mass moving with unit velocity.

33. Newton's Laws of Motion.

Law I. *Every body will continue in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by impressed force to change that state.*

Law II. *The rate of change of momentum is proportional to the impressed force, and takes place in the direction in which that force acts.*

Law III. *To every action there is an equal and opposite reaction.*

34. These laws cannot be proved experimentally or otherwise, but our observations of the phenomena of nature lead us to infer their truth. For instance, working on the assumption of the truth of these laws Adams and Le Verrier discovered the path and position of the planet Neptune before any human eye had seen it: indeed the whole theory of Astronomy is based upon the truth of these laws. With their aid, in combination with the law of gravitation, the exact position of the moon at any instant, the exact times of eclipses, the exact time of high-tide at any port, can all be predicted with absolute certainty.

35. **Law I.** We have no opportunity of observing a body under the action of *no* forces. If a ball be rolled along

a horizontal plane, it at length comes to rest; but the smoother the plane and the ball, the further the ball will roll before stopping, and we therefore infer that if the motion were *in vacuo* and the ball and plane perfectly smooth the ball would continue to move with uniform velocity for ever.

This law involves the property of matter commonly called the *Principle of Inertia*, the principle that a body has no tendency in itself to change its state of rest or uniform motion in a straight line. Thus if a body be at rest or moving uniformly in a straight line it will remain so unless some external force act upon it.

When we put coal on the fire, the hand stays the shovel, but there being no corresponding force acting on the coal, it precipitates itself into the grate.

In the case of a bicycle-rider whose bicycle is suddenly stopped, the impeding force does not act on the man, his motion therefore continues, i.e. he flies over the handles.

36. Law II. "Rate of change of momentum" here must be taken to mean the change of momentum in the unit of time.

Let m be the mass of the body, f its acceleration, and let a force P act upon it.

Then by this law,

$P \propto$ the rate of change of momentum,

$\propto$ the rate of change of mv,

$\propto m \times$ the rate of change of v (m being unchanged),

$\propto m\,.f.$ (Def. of acceleration.)

$\therefore\ P = kmf$, where k is some constant quantity.

Now if we take the unit of force as that force which acting on the unit of mass produces the unit of acceleration,

$$\text{when } m = 1, \text{ and } f = 1, \text{ we have } P = 1,$$

$$\therefore\ k = 1.$$

Hence we have the formula

$$P = mf.$$

When a pound, a foot, and a second are taken as units of mass, length, and time respectively, the unit of force is called a *Poundal.*

Hence if a force of P poundals act on a mass m lbs. and produce an acceleration of f ft.-sec. units,

$$\mathbf{P = mf.}$$

This may also be expressed thus:—

$$\text{Acceleration} = \frac{\text{moving force}}{\text{mass moved}}. \qquad \left(f = \frac{P}{m}\right)$$

37. From this law we also deduce what is usually called the principle of the **Physical Independence of Forces.**

The latter part of the law asserts that the change of momentum takes place in *the direction in which the force acts.* Thus if two forces act upon a body, each produces its own effect in its own direction, independently of the other, for the second law is true for each of these forces.

It follows that if a number of forces act upon a body the equation $P = mf$ is true for each.

Hence if a shot be fired *horizontally* with any velocity from the top of a cliff of height h, and strike the sea in time t, it will be found that $h = \frac{1}{2}gt^2$; and its vertical velocity on striking the sea will be $\sqrt{2gh}$, i.e. the effect of gravity, being vertical and therefore at right angles to the velocity of projection, is quite independent of that velocity, and the time of falling and the vertical velocity on striking the sea are the same as if no horizontal initial velocity had been given to the shot.

We also infer from the second law that since a force has its full effect in its own direction, *it has no effect in a direction at right angles to its own.*

38. *The relation between the unit of force and the weight of the unit of mass.*

When a body falls freely *in vacuo,* the only force acting upon it is its own weight, and we know that this produces an acceleration g.

Hence the wt. of 1 lb. acting on the mass of 1 lb. produces acceleration g.

$\therefore$ the wt. of $\frac{1}{g}$ lb. acting on the mass of 1 lb. produces the unit of acceleration.

But the unit of force acting on the unit of mass produces the unit of acceleration.

$\therefore$ if we take the mass of 1 lb. as the unit of mass we must take $\frac{1}{g}$ lb. wt. as the unit of force,

i.e. $\qquad$ **$\frac{1}{g}$ lb. wt. = a poundal,**

or $\qquad$ **g poundals = 1 lb. wt.**

N.B. If W poundals be the weight of m lbs., since the weight of a body acting on its own mass produces acceleration g,

$$W = mg.$$

39. Since the value of g varies at different places on the earth's surface the weight of 1 lb., i.e. g poundals, is not a constant quantity. On the other hand a single poundal is constant in value, and is therefore termed *an absolute unit.*

40. The scientific (French) unit of mass is called a *gramme.* It was originally meant to be the mass of a cubic centimetre of pure water at a temperature of 4° Centigrade.

When a gramme, a centimetre, and a second are taken respectively as the units of mass, length, and time, the corresponding unit of force is termed a **Dyne.**

Thus if a force of P dynes act on a mass of m grammes and produce acceleration f centimetre-sec. units,

$$P = mf.$$

Remembering therefore that $g = 981$ centimetre-sec. units, we see that the weight of one gramme $= g$ dynes.

Here again we must notice that, g being a variable quantity, a gramme weight ($= g$ dynes) is also variable, whereas a dyne is constant in value, and therefore *an absolute unit.*

41. Ex. i. *A mass of 10 tons initially at rest is acted upon by a force constant in magnitude and direction, and equal to the weight of 14 lbs. After what interval will it have a velocity of 1 foot per second?*

Let f be the acceleration produced by the force of 10 lbs. wt., and t seconds the time required.

Then since 14 lbs. wt. $=14g=14\times 32$ poundals,

$$14\times 32=10\times 2240f. \qquad (P=mf)$$

[N.B. We express the force in poundals and the mass in pounds.]

Whence $f=\frac{1}{50}$ ft.-sec. units.

$$\therefore\ 1=\tfrac{1}{50}t, \qquad (v=u+ft)$$

whence $t=50$ seconds.

Ex. ii. *A constant force acts upon a mass of 4 lbs. during 3 secs. from rest, and then ceases; in the next 3 seconds it is found that the mass describes 72 feet. Find the magnitude of the force.*

Let P poundals be the force required, f the acceleration produced by it, and v the velocity of the mass when the force ceases to act.

After the force has ceased to act, the mass will move with uniform velocity,

$$\therefore\ 72=3v, \qquad (s=ut)$$

i.e. $v=24$ ft. per sec.

Hence, for the motion during the first 3 seconds,

$$24=3f. \qquad (v=u+ft)$$

$$\therefore f=8 \text{ ft.-sec. units.}$$

$$\therefore P=4\times 8 \text{ poundals} \qquad (P=mf)$$

$$=1 \text{ lb. wt.}$$

Ex. iii. *A body of mass 3 lbs. has been falling freely under the action of gravity for 4 seconds; find what vertical force applied to it will bring it to rest in 64 feet.*

Let P poundals be the force required, and f the retardation produced when this force acts on the mass.

The velocity acquired in its fall during 4 seconds

$$=4g \text{ feet per second.} \qquad (v=u+ft)$$

Hence for the motion when P is acting

$$0=(4g)^2-2f\,.\,64. \qquad (v^2=u^2+2fs)$$

$$\therefore f=128 \text{ ft.-sec. units.}$$

The resultant force acting upon the mass is $(P-3g)$ poundals.

[For we have P poundals acting upon the mass vertically upwards and the weight of the mass, $3g$ poundals, acting vertically downwards.]

$$\therefore P-3g=3f, \qquad (P=mf)$$

i.e.
$$P=3(g+f)$$
$$=3(32+128) \text{ poundals}$$
$$=15 \text{ lbs. wt.}$$

Throughout this book all forces may be taken as expressed in poundals, and all masses in pounds: e.g. a force P means a force of P poundals, and a mass m, a mass of m lbs.

EXAMPLES. IV.

[In all examples, if foot-second units be used the British standard units of mass and force must be used, i.e. mass must be expressed in pounds, and force in poundals. If the scientific (C.G.S.) units be adopted, mass must be expressed in grammes and force in dynes.]

1. A force of 6 poundals acts horizontally upon a mass of 3 pounds initially at rest; find the acceleration produced, and the space traversed in 4 seconds.

2. A force of 4 poundals acts horizontally upon a mass of 3 lbs. which has initially a velocity of 3 ft. per sec.; find the acceleration generated, and the velocity of the mass in 6 seconds.

3. A mass of 16 lbs. is acted upon in a horizontal direction by a force of 2 lbs. wt. for 20 seconds; find the acceleration generated, and the space traversed if the body started with a velocity of 10 feet per second.

4. What force acting upon a mass of 24 lbs. in a horizontal direction will generate an acceleration of 16 ft.-sec. units? Express the force in poundals, and in pounds wt.

5. A force of 4 lbs. wt. acting upon a body in a horizontal direction generates in it an acceleration of 8 ft.-sec. units; find the mass of the body.

6. A force of 1 ton wt. acts upon a mass of 1 cwt. in a horizontal direction for one second; find the velocity generated. If the force then ceases to act, how far will the mass move in the next two seconds?

7. In what time will a force of 3 lbs. wt. acting horizontally upon a mass of 6 lbs. generate in it a velocity of 64 feet per second?

8. A force equal to the weight of 1 cwt. acts horizontally on a mass of one ton for half a minute; find the velocity generated, and the space traversed.

9. A body, acted upon by a uniform force, moves through 100 feet in 10 seconds from rest; find the ratio of the force to the weight of the body.

10. Find the force which acting on a mass of 36 lbs. will generate in 11 seconds a velocity of 20 miles an hour.

11. A force of 48 poundals acts horizontally upon a mass of 12 lbs., initially at rest, for 3 seconds, and then ceases: how far will the mass move in the next 4 seconds?

12. A particle of mass 2 lbs. initially at rest, when acted upon by a constant force moves over 18 feet in the fifth second of its motion: find the force.

13. A particle acted on by a force of 90 dynes passes over 36 centimetres whilst its velocity increases from 5 to 13 centimetres per second: find its mass.

14. A force of 24 dynes acts horizontally upon a mass of 4 grammes; how long will it take to generate a velocity of 36 centimetres per second?

15. A force equal to the weight of 20 grammes acting upon a particle for 10 seconds moves it from rest through 10 metres; find the mass of the particle.

16. A mass of m lbs. is acted on by a force of p lbs. wt. for t minutes; find the velocity generated, and the space traversed from rest.

17. How long will a force equal to the weight of 1 kilogramme, acting upon a mass of 109 grammes, take to generate a velocity of 900 metres per second?

18. A train of 80 tons, at rest on a horizontal plane, is acted upon by a force of 1000 lbs. wt.; the force acts for 8 minutes and then ceases; how much further will the train travel before again coming to rest, supposing the friction of the rails to be equivalent to a retarding force of 300 lbs. wt.?

19. A bullet leaves the muzzle of a rifle with a velocity of 1280 feet per second; if the barrel of the rifle is 4 feet long, and the mass of the bullet 2 ozs., find the force (supposed uniform) acting on the bullet whilst it is in the barrel.

20. The weight of a train is 100 tons, the resistance arising from friction etc. 7 lbs. wt. per ton. Find the acceleration when the tractive force acting upon it is equal to the weight of half a ton.

21. A heavy vertical chain is drawn upwards by a given force P, which exceeds its weight W. Find its acceleration, and its tension at any assigned point.

22. Each of two bodies at rest attracts the other with the same force. If allowed to move, shew that in the first instant of motion they move over spaces which are inversely as their masses.

CHAPTER V.

PARALLELOGRAM OF VELOCITIES.

42. HITHERTO we have treated of bodies having but one velocity; let us now consider bodies possessing simultaneously two or more velocities. A simple illustration of motion of this kind is when a vessel steams in one direction, whilst a current drifts it in another: this ship has two velocities, the velocity with which it steams in one direction, and the velocity with which it is carried by the current in another.

Again, a point on the surface of the earth has two velocities in space; it moves with the earth in its daily rotatory motion about its axis, and it also moves with the earth in its path round the sun.

Velocities may be represented by straight lines.

Velocities have magnitude and direction. Hence since a straight line can be drawn of any length, and in any direction, we may represent velocities by straight lines.

Thus if we take a straight line AB to represent a velocity u of a point, we mean that the point moves in the direction AB, and that the length of AB is proportional to u.

43. *That velocity which is equivalent to two or more velocities is called their* **resultant**; *and these velocities are called the* **components** *of this resultant.*

Also when we substitute for a velocity its components in any directions we are said to **resolve** that velocity into its components in those directions.

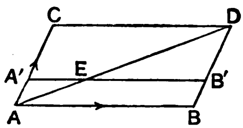
44. **Parallelogram of Velocities.** *If a moving point possess simultaneously velocities which are represented in*

magnitude and direction by the two sides of a parallelogram drawn from a point, they are equivalent to a velocity which is represented in magnitude and direction by the diagonal of the parallelogram drawn from that point.

Let the two straight lines AB, AC represent two coexisting velocities u and v of a point: complete the parallelogram $ABDC$.

Then the diagonal AD will represent the resultant velocity of the point in magnitude and direction.

The coexistence of the two velocities u and v may be represented by supposing the point to move uniformly along AB with velocity u, whilst AB moves parallel to itself with uniform velocity v, its ends moving along AC and BD.



Then in the unit of time the point moves along AB from A to B, whilst AB travels parallel to itself into the position CD.

Suppose that, at time t, AB has reached the position $A'B'$, and the moving point has reached E.

Then since $AA' = vt$, and $A'E = ut$, $\qquad (s = ut)$

$$AA' : A'E :: v : u,$$
$$:: AC : CD.$$

But the angle $AA'E =$ angle ACD.

$\therefore$ the triangles $AA'E$, ACD are similar,

i.e. the angle $A'AE =$ the angle CAD.

$\therefore$ E must lie in AD.

$\therefore$ the point E travels *along* AD.

Again, since the triangle $A'AE$ remains similar to triangle CAD, AE is proportional to AA', which increases uniformly.

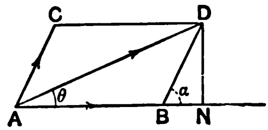
$\therefore$ the point E travels *uniformly* along AD.

Hence in the unit of time the point travels uniformly from A to D.

$\therefore$ AD represents its resultant velocity.

45. *To find the magnitude and direction of the resultant of two velocities u and v inclined at an angle α to one another.*

Let AB and AC represent the velocities u and v respectively. Complete the parallelogram $ABDC$. Join DA, and let θ denote the angle DAB. By the Parallelogram of Velocities, AD represents the resultant.



In the triangle ABD

$$AD^2 = AB^2 + BD^2 - 2 \, . \, AB \, . \, BD \cos ABD$$

$$(a^2 = b^2 + c^2 - 2bc \cos A),$$

i.e.

$$AD^2 = u^2 + v^2 - 2uv \cos(\pi - \alpha)$$

$$(\angle ABD = \pi - \alpha),$$

$$= u^2 + v^2 + 2uv \cos \alpha,$$

$$\therefore \; AD = \sqrt{u^2 + v^2 + 2uv \cos \alpha}.$$

Draw DN at right angles to AB,

$$\tan\theta = \frac{DN}{AN} = \frac{DN}{AB + BN}$$

$$= \frac{v \sin \alpha}{u + v \cos \alpha}.$$

$$\therefore \; \theta = \tan^{-1}\left(\frac{v \sin \alpha}{u + v \cos \alpha}\right).$$

Alternative method for finding the angle θ.

$$\angle ADB = \alpha - \theta.$$

$\therefore$ in the triangle ADB $\quad \dfrac{\sin(\alpha - \theta)}{\sin\theta} = \dfrac{u}{v},$

i.e.

$$\frac{\sin \alpha \cos \theta - \cos \alpha \sin \theta}{\sin \theta} = \frac{u}{v},$$

$$\sin \alpha \cot \theta - \cos \alpha = \frac{u}{v},$$

places. Thus if a body be weighed by means of a spring balance, first at a point on the equator, and then at some point north or south of the equator, its weight in this latter position will be found to be greater than at the equator—it being nearer to the centre of the earth.

Or again, if we imagine a body removed to an infinite distance from any other body its mass (i.e. the quantity of matter in it) would remain unaltered but its weight would be nil.

Momentum. The *momentum*, or quantity of motion of a particle, is the product of its mass and its velocity. It is denoted by mv where m is the mass and v the velocity of the particle.

The unit of momentum is therefore the momentum of the unit of mass moving with unit velocity.

33. Newton's Laws of Motion.

Law I. *Every body will continue in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by impressed force to change that state.*

Law II. *The rate of change of momentum is proportional to the impressed force, and takes place in the direction in which that force acts.*

Law III. *To every action there is an equal and opposite reaction.*

34. These laws cannot be proved experimentally or otherwise, but our observations of the phenomena of nature lead us to infer their truth. For instance, working on the assumption of the truth of these laws Adams and Le Verrier discovered the path and position of the planet Neptune before any human eye had seen it: indeed the whole theory of Astronomy is based upon the truth of these laws. With their aid, in combination with the law of gravitation, the exact position of the moon at any instant, the exact times of eclipses, the exact time of high-tide at any port, can all be predicted with absolute certainty.

35. **Law I.** We have no opportunity of observing a body under the action of *no* forces. If a ball be rolled along

a horizontal plane, it at length comes to rest; but the smoother the plane and the ball, the further the ball will roll before stopping, and we therefore infer that if the motion were *in vacuo* and the ball and plane perfectly smooth the ball would continue to move with uniform velocity for ever.

This law involves the property of matter commonly called the *Principle of Inertia*, the principle that a body has no tendency in itself to change its state of rest or uniform motion in a straight line. Thus if a body be at rest or moving uniformly in a straight line it will remain so unless some external force act upon it.

When we put coal on the fire, the hand stays the shovel, but there being no corresponding force acting on the coal, it precipitates itself into the grate.

In the case of a bicycle-rider whose bicycle is suddenly stopped, the impeding force does not act on the man, his motion therefore continues, i.e. he flies over the handles.

36. Law II. "Rate of change of momentum" here must be taken to mean the change of momentum in the unit of time.

Let m be the mass of the body, f its acceleration, and let a force P act upon it.

Then by this law,

$$\begin{aligned} P &\propto \text{the rate of change of momentum,} \\ &\propto \text{the rate of change of } mv, \\ &\propto m \times \text{the rate of change of } v \ (m \text{ being unchanged}), \\ &\propto m \, . f. \qquad\qquad \text{(Def. of acceleration.)} \end{aligned}$$

$\therefore P = kmf$, where k is some constant quantity.

Now if we take the unit of force as that force which acting on the unit of mass produces the unit of acceleration,

$$\text{when } m = 1, \text{ and } f = 1, \text{ we have } P = 1,$$

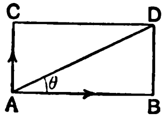
$$\therefore k = 1.$$

Hence we have the formula

$$P = mf.$$

whence
$$\cot\theta = \frac{u + v\cos\alpha}{v\sin\alpha},$$
$$\tan\theta = \frac{v\sin\alpha}{u + v\cos\alpha},$$
$$\theta = \tan^{-1}\left(\frac{v\sin\alpha}{u + v\cos\alpha}\right).$$

COR. 1. If u and v be at right angles



$$AD^2 = AB^2 + BD^2 = u^2 + v^2,$$
$$\therefore\ AD = \sqrt{u^2 + v^2}$$
and
$$\tan\theta = \frac{BD}{AB} = \frac{v}{u},$$
$$\therefore\ \theta = \tan^{-1}\left(\frac{v}{u}\right).$$

COR. 2. Again if u and v be at right angles, and V denote their resultant, we see from the figure
$$u = V\cos\theta,$$
$$v = V\sin\theta.$$

Hence, the following important theorem :—**If the direction of a velocity V make an angle θ with any line, this velocity is equivalent to a velocity V cos θ along that line, together with a velocity V sin θ at right angles to that line.**

46. *The resolved part of a given velocity in any direction is that velocity which represents the whole effect of the given velocity in that direction.*

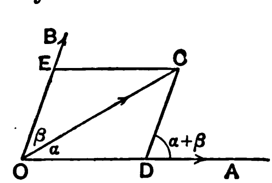
Thus in Cor. 1 of the previous article, the velocity AD is equivalent to velocities AB and AC, and the velocity AC, being perpendicular to AB, has no effect in that direction, therefore AB represents the whole effect of the velocity AD along AB, and is therefore its resolved part in that direction.

Similarly AC represents its resolved part in the direction AC.

Hence if a velocity V in a direction AB make an angle θ with any line AC, the resolved part of V in the direction AC is $V\cos\theta$; and its resolved part perpendicular to AC is $V\sin\theta$.

47. *To find the components of a given velocity in two given directions.*

Let OC represent the given velocity u. Draw OA and OB making angles α and β with OC. It is required to find the components of the velocity u in the directions OA and OB.



From C draw CD parallel to OB to meet OA in D, and CE parallel to OD to meet OB in E.

Then by the Parallelogram of Velocities OD and OE are the components required.

Also from the triangle ODC

$$\frac{OD}{\sin OCD}=\frac{CD}{\sin COD}=\frac{OC}{\sin ODC}.$$

$$\therefore\quad \frac{OD}{\sin\beta}=\frac{CD}{\sin\alpha}=\frac{u}{\sin(\alpha+\beta)},$$

i.e.
$$OD=\frac{u\sin\beta}{\sin(\alpha+\beta)},$$

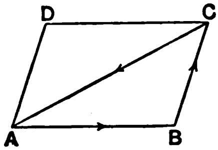
and
$$OE=CD=\frac{u\sin\alpha}{\sin(\alpha+\beta)}.$$

48. Triangle of Velocities. *If a particle possess simultaneously three velocities represented in magnitude and direction by the three sides of a triangle, taken in order, the particle is at rest.*

Let a particle possess simultaneously three velocities represented by the sides AB, BC, CA, taken in order, of the triangle ABC.

Complete the parallelogram $ABCD$.

Then since $BC = AD$ and is parallel to it, the resultant of the velocities AB, BC is equal to that of the velocities AB, AD and is therefore represented by AC.



(Parallelogram of Velocities.)

$\therefore$ the resultant of the velocities AB, BC, CA

= the resultant of the velocities AC, CA

= zero.

$\therefore$ the particle is at rest.

COR. **Note that the resultant of velocities AB, BC is represented by AC.**

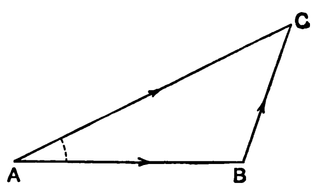
49. *A particle has simultaneously two given velocities in given directions: to determine graphically the magnitude and direction of its resultant velocity.*

Draw AB and BC in the given directions to represent the given velocities.

Then by the Triangle of Velocities, AC represents the resultant velocity in magnitude and direction.

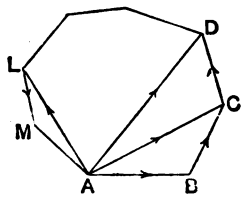
The magnitude of AC and the $\angle CAB$ are obtained by actual measurement.

These determine the magnitude and direction of the resultant velocity.



50. Polygon of Velocities. *If a particle possess simultaneously velocities represented in magnitude and direction by the sides of a polygon, taken in order, the particle is at rest.*

Let $ABC...LM$ be the polygon, then by the Triangle of Velocities, the resultant of velocities AB, BC is represented by AC.



The resultant of velocities AC, CD is represented by AD, and so on until we come to LM, when we see the resultant velocity of the system is represented by AM and MA, i.e. it is zero. The particle is therefore at rest.

51. Ex. i. *A boat is rowed across a stream at right angles to its banks with a velocity of 4 miles an hour, and drifts with the current which runs at the rate of 3 miles an hour. Find the resultant velocity of the boat.*

Let V denote its resultant velocity, making an angle θ with the bank.

Then $V\cos\theta$ = its velocity in the direction of the current

$$=3 \text{ m. an hour},$$

and $V\sin\theta$ = its velocity in the direction at right angles to the bank

$$=4 \text{ m. an hour};$$

$$\therefore\ V^2=3^2+4^2=25,$$

$$V=5 \text{ m. an hour},$$

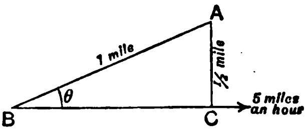
and $$\tan\theta=\tfrac{4}{3},\ \theta=\tan^{-1}(\tfrac{4}{3}),$$

i.e. the resultant velocity is 5 miles an hour making an angle $\tan^{-1}(\frac{4}{3})$ with the bank.

Ex. ii. *A person walking along a road at 5 miles per hour, sees a tower a mile distant from his eye, the nearest distance of the tower from the road being half a mile. Find the rate at which he is approaching the tower. Does this rate alter as he advances?*

Let A be the tower, BC the road, so that BA = one mile, and $CA = \frac{1}{2}$ mile, where CA is perpendicular to BC.

Let $\angle ABC = \theta$.



The rate at which the man approaches the tower

$$= \text{his velocity in the direction } BA$$

$$= 5 \cos\theta \text{ miles per hour}$$

$$= \frac{5\sqrt{3}}{2} \text{ miles per hour (for } BC^2 = 1 - \tfrac{1}{4} = \tfrac{3}{4}\text{)}.$$

As the man advances the angle θ alters, whilst his velocity along the road remains constant.

Therefore his velocity in the direction BA ($5 \cos\theta$ miles per hour) changes. We notice that from B to C the angle θ increases, therefore its cosine diminishes and the velocity of approach diminishes with it. After the man has passed the point C his velocity of approach becomes negative and increases numerically.

EXAMPLES. V.

[Examples marked with an asterisk (*) may be solved graphically, or by calculation. A good plan is to solve by calculation, and verify by the graphical method.]

[Use Mathematical Tables where necessary.]

*1. A ship is steaming due north at $4\sqrt{3}$ miles an hour, and a man walks across its deck in a direction due west at 4 miles an hour: find his resultant velocity, and direction in space.

*2. A balloon rises vertically with a velocity of 24 ft. per second, and drifts horizontally with the wind with a velocity of 10 ft. per second. Find the magnitude and direction of its resultant velocity.

3. A man points and rows his boat straight across a stream a quarter of a mile broad; he finds that he reaches the opposite bank 220 yards below the point opposite his starting point: find the velocity of the current if the man rows 6 miles an hour.

*4. A steamer leaves a pier and steams north at 10 miles an hour, whilst the current runs west at 5 miles an hour. Find the distance of the steamer from the pier after 2 hours.

*5. A particle has two velocities of 3 ft. per sec. and 5 ft. per sec. at an angle of 120° with one another. Find the magnitude of its resultant velocity.

6. A body has two simultaneous velocities of 7 and 5 ft. per sec. at an angle of 50° with one another. Find graphically its resultant velocity in magnitude and direction.

7. A man points his boat straight across a stream and rows it with a velocity of 5 miles per hour. If the stream runs at the rate of 3 miles per hour, find the direction of the course of the boat, and its resultant velocity. Verify your result graphically.

8. A point moves in a straight line with a velocity of 3 ft. per sec. After 2 secs. it has an additional velocity of 4 ft. per sec. at right angles to its original motion: find its distance from the starting point 2 secs. after this, and verify your result graphically.

9. A man keeps his boat at right angles to the current and rows it across a stream 50 yds. wide in one minute. On landing he finds himself 60 yds. from his starting point. Find the velocity of the current, and verify your result graphically.

*10. Resolve a velocity of $4\sqrt{3}$ ft. per second into components making angles of 30° and 90° with its direction.

*11. A man who swims in a direction 30° east of north with a velocity of 3 miles an hour, finds that a westerly current causes his actual direction in space to be due north. Find the velocity of the current.

*12. Resolve a velocity v into two components each making an angle of 30° with its direction.

*13. Find the distance over which a particle moves in 10 secs. when it has a velocity of 4 ft. per sec. northwards and 3 ft. per sec. eastwards.

*14. Two ships sail from the same point eastwards and north-eastwards with velocities of 6 and $6\sqrt{2}$ miles an hour respectively: how far apart are they in 2 hours?

*15. A point has two velocities v and $2v$ ft. per sec. at an angle of 120° with one another. How far is it from its starting point in 4 seconds?

*16. A velocity of 9 ft. per sec. has one component making an angle of 60° with it, and equal to 3 ft. per sec.: find the magnitude and direction of the other component.

*17. A point possesses simultaneously velocities of 2, 3, 8 ft. per second making angles of 120° with one another: find the magnitude of its resultant velocity.

18. A point possesses simultaneously velocities, each equal to v, acting from the centre of a regular pentagon to four of its angular points: find the magnitude and direction of its resultant velocity.

*19. A ball moves horizontally with a velocity of 6 ft. per sec. After 2 secs. it has communicated to it an additional velocity of 5 ft. per sec. at right angles to its original velocity: find its distance from the starting point 2 seconds after this.

20. A body descends uniformly down an inclined plane 2 miles in length in 4 minutes, find its vertical velocity in ft.-sec. units if the plane rises 1 in 20.

21. A stream has a current of velocity v, and a man can row his boat with a velocity $v\sqrt{2}$; in what direction must he row across if he is to land at a point exactly opposite his starting point?

22. A man rows his boat across a stream, always pointing it at right angles to the banks: if he rows at the same speed, he crosses the stream in the *same time* whatever the velocity of the current. Explain this.

23. A man rows his boat across a stream a mile broad, always pointing his boat up stream at an angle of 30° with the bank; how long does he take to cross if he rows with a velocity of 4 miles an hour?

*24. A point possesses simultaneously velocities u, $2u$, $3u$, $4u$, in directions north, east, south, and west respectively: find its resultant velocity.

25. A point has velocities represented by AB, AC two sides of a triangle; shew that its resultant velocity is represented by $2AD$, where D is the middle point of BC.

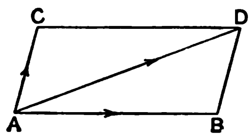
26. A point has two velocities, one of fifteen miles an hour northwards, and the other of $16\frac{1}{2}$ ft. per sec. eastwards; how long will the point take to travel 100 miles?

CHAPTER VI.

PARALLELOGRAM OF ACCELERATIONS, ETC.

52. **Parallelogram of Accelerations.** *If a moving point possess simultaneously two accelerations represented in magnitude and direction by two sides of a parallelogram drawn from a point, their resultant acceleration is represented in magnitude and direction by the diagonal of the parallelogram drawn from that point.*

Let AB, AC represent two coexisting accelerations; complete the parallelogram $ABDC$. Then AD shall represent their resultant acceleration in magnitude and direction.



By definition, AB and AC represent the velocities added in the unit of time.

Therefore the diagonal AD represents the resultant velocity added in the unit of time.

(Parallelogram of Velocities.)

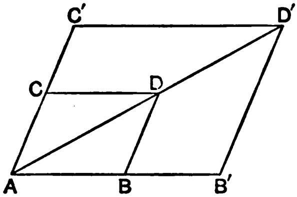
i.e. AD represents the resultant acceleration.

It follows from the above that accelerations may be resolved and compounded in the same way as velocities; and hence we can prove, in the same way as for velocities, the "*Triangle of Accelerations,*" and the "*Polygon of Accelerations.*"

53. **Parallelogram of Forces.** *If two forces acting on a body be represented in magnitude and direction by two sides*

of a parallelogram drawn from a point, their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn from that point.

Let P_1, P_2 poundals be the two forces acting in directions AB, AC.



By the principle of the Physical Independence of Forces, each force will produce an acceleration in its own direction independently of the other. Let f_1, f_2, be these accelerations, and take AB and AC to represent them. Let m lbs. be the mass of the body upon which the forces act, f the resultant acceleration of f_1 and f_2; P the resultant force, so that $P = mf$.

Then $\qquad P_1 = mf_1 = mAB$ and $P_2 = mf_2 = mAC$.

Take AB' to represent the force P_1, and AC' to represent the force P_2.

Complete the parallelograms $ABDC$, $AB'D'C'$.

By the Parallelogram of Accelerations AD represents f.

Now $$\frac{AC'}{C'D'} = \frac{P_2}{P_1} = \frac{mf_2}{mf_1} = \frac{AC}{CD},$$

and $$\angle\, ACD = \angle\, AC'D'.$$

$\therefore$ Δs $C'AD'$, CAD are similar,

i.e. $$\angle\, C'AD' = \angle\, CAD,$$

and hence ADD' is a straight line.

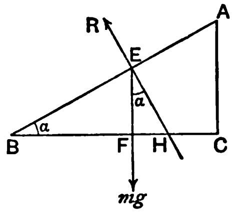
Also $$\frac{AD'}{AD} = \frac{AC'}{AC} = \frac{mf_2}{f_2} = m.$$

Therefore since $AD = f$, $AD' = mf = P$ the resultant of forces P_1 and P_2, i.e. AD' represents the resultant of the forces AC' and AB'.

Hence we see that *forces* may be resolved and compounded in the same way as velocities and accelerations; and we therefore have a "*Triangle of Forces*," and a "*Polygon of Forces*."

54. Motion on a smooth inclined plane. Suppose a body of mass m to move freely upon an inclined plane.

Let AB be the line of greatest slope of the plane whose elevation is α to the horizon, the figure thus being a vertical section.



The forces acting on the body at any point E on the plane are:—

(1) its own weight mg vertically downwards;

(2) the resistance of the plane (R) at right angles to the plane.

['mg' may be resolved into $mg\cos\alpha$ perpendicular to the plane, and $mg\sin\alpha$ down the plane.]

There is no motion at right angles to the plane, therefore resolving in that direction

$$R - mg\cos\alpha = 0.$$

The remaining component of mg, viz. $mg\sin\alpha$ in the direction EB, acts upon the body in that direction and therefore produces an acceleration f, where

$$mg\sin\alpha = mf, \qquad (P = mf)$$

i.e.

$$f = g\sin\alpha.$$

Hence a body allowed to move freely upon a smooth inclined plane of elevation α, will be subject to an acceleration $g\sin\alpha$ down the plane in the direction of greatest slope.

55. It follows from this that if a body, projected *down* a plane with a velocity u, describes space s along the plane in time t, and acquires velocity v:—

$$v = u + g\sin\alpha \,.\, t, \qquad (v = u + ft)$$

$$s = ut + \tfrac{1}{2}g\sin\alpha \,.\, t^2, \qquad (s = ut + \tfrac{1}{2}ft^2)$$

$$v^2 = u^2 + 2g\sin\alpha \,.\, s. \qquad (v^2 = u^2 + 2fs)$$

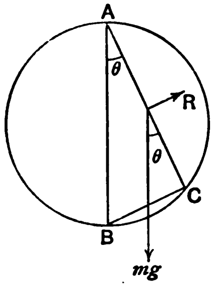
Also if the body be projected up the plane these formulae become

$$v = u - g \sin \alpha \,.\, t,$$

$$s = ut - \tfrac{1}{2} g \sin \alpha \,.\, t^2,$$

$$v^2 = u^2 - 2g \sin \alpha \,.\, s.$$

56. *A heavy particle slides from rest at the highest point of a vertical circle down a chord of the circle: to shew that its time of descent is constant, i.e. the same for all chords.*



Let m be the mass of the particle, θ the inclination of the chord AC to the vertical, AB the vertical diameter ($=2a$).

The particle is subject to an acceleration $g \cos \theta$ in direction AC by the preceding article, therefore if t be the time of descent,

$$AC = \tfrac{1}{2} g \cos \theta \,.\, t^2, \qquad (s = ut + \tfrac{1}{2} ft^2)$$

i.e. $$2a \cos \theta = \tfrac{1}{2} g \cos \theta \,.\, t^2,$$

$$\therefore \; t^2 = \frac{4a}{g},$$

$$t = 2\sqrt{\frac{a}{g}},$$

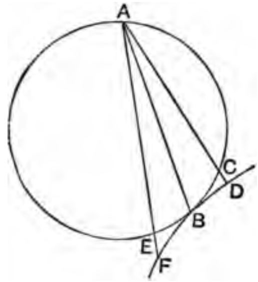
which is independent of θ, and therefore constant.

Cor. *In the same way it may be proved that the time of descent down all chords of a vertical circle drawn to the lowest point is constant.*

57. **Line of quickest descent.** *To draw a straight line from a given point to a given curve in the same vertical plane, so that a body may slide down it to the curve in the shortest possible time.*

Let A be the given point, DBF the given curve. Describe

a circle having its highest point at A and touching the given curve at B. AB shall be the line of quickest descent.



For consider any other chord from A meeting the circle at C, and the curve at D. By the preceding article,

$$\text{the time down } AB = \text{the time down } AC$$
$$< \text{the time down } AD, \text{ for } AD > AC.$$

Hence the time down AB is less than the time down any other chord from the point A to the curve, i.e.

AB is the line of quickest descent.

COR. To find the line of quickest descent from a given curve to a given point A, describe a circle having its lowest point at A, and touching the curve at B. It may then be similarly proved that BA is the line of quickest descent.

58. Ex. i. *A body is projected up an inclined plane of elevation* 30° *and length* 384 *yards: find its initial velocity if it just reaches the top.*

Let u be the initial velocity.

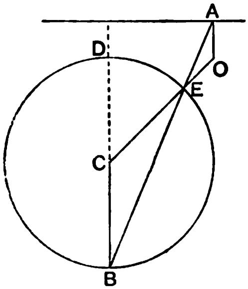
When the body reaches the top its velocity is zero, and it is subject to an acceleration $g \sin 30°$ down the plane;

$$\therefore\ 0 = u^2 - 2g \sin 30° . 384 . 3. \qquad (v^2 = u^2 + 2fs)$$
$$\therefore\ u^2 = 32 . 3 . 384,$$
$$u = 192 \text{ ft. per second.}$$

Ex. ii. *To find the line of quickest descent from a given point to a given circle in the same vertical plane.*

Let A be the given point, DEB the given circle. Take C the centre of the circle and draw a vertical radius to meet the circle in its lowest point B. Join AB, cutting the circle at E. AE shall be the line of quickest descent.

Join CE, and produce it to meet the vertical through A at O.



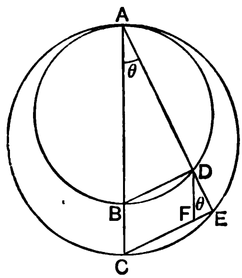
$$\begin{aligned} \angle OAE &= \angle CBE & &\text{(by parallels)} \\ &= \angle CEB & &\text{(for } CE = CB\text{)} \\ &= \angle AEO. \end{aligned}$$

$$\therefore\ OE = OA,$$

i.e. a circle described with centre O and radius OA will have A for its highest point, and will touch the given circle at E.

$\therefore$ AE is the line of quickest descent.

Ex. iii. *If two vertical circles touch at their highest points, and a straight line be drawn from this point cutting the circles, shew that the time from rest down the part between the circumferences is constant.*



Let A be the highest point of the circles, ADE a chord through A, ABC the common vertical diameter, θ the inclination of AD to the vertical.

Join BD, CE; and draw DF the vertical through D.

Since $\angle ADB$ = a right angle = $\angle AEC$, BD is parallel to CE.

Hence since DF is parallel to BC, it is also equal to it.

Let t be the time down DE; the acceleration down $DE = g\cos\theta$.

$$DE = \tfrac{1}{2}g\cos\theta \,.\, t^2. \qquad (s = ut + \tfrac{1}{2}ft^2)$$

$$\therefore\ t^2 = \frac{2DE}{g\cos\theta} = \frac{2DF\cos\theta}{g\cos\theta} = \frac{2BC}{g}.$$

$$\therefore\ t = \sqrt{\frac{2BC}{g}} \text{ and is therefore constant.}$$

EXAMPLES. VI.

1. A body slides from rest down an inclined plane of elevation 30°; find how far it has gone, and its velocity, in 3 seconds.

2. A particle describes $200\sqrt{2}$ feet from rest sliding down an inclined plane in 5 seconds. Find the inclination of the plane.

3. A heavy particle projected up an inclined plane with a velocity of 64 feet per second, describes $128(2-\sqrt{3})$ feet in 4 seconds. Find the inclination of the plane.

4. A body slides from rest down an inclined plane of elevation 30°; find the distance in centimetres passed over in the fourth second of its motion.

5. In what time would a body slide down 1000 yds. of an inclined plane rising 1 in 30?

6. A particle slides down an inclined plane 120 yards long in 5 seconds: find the height of the plane.

7. A particle slides down an inclined plane 64 feet long inclined to the horizon at 30°: find the time of describing the lower half.

8. A particle projected down an inclined plane of length l, and height h, reaches the bottom simultaneously with another particle which starts at the same instant and falls freely through the vertical height of the plane: find its initial velocity.

9. Prove that the velocity acquired by a body in sliding down an inclined plane is equal to that acquired by falling freely through the vertical height of the plane.

Find the line of quickest descent:—

10. To a given straight line from a given point in the same vertical plane.

11. From a given straight line to a given point in the same vertical plane.

12. From a given circle to a given point without it, the point and the circle being in the same vertical plane.

13. From a given vertical circle to a given point within it.

14. From a given point within a given vertical circle to the circle.

15. From a given vertical circle to a given vertical straight line without it, and in its plane.

16. From a given vertical straight line to a given circle in the same vertical plane.

17. The time of sliding down a smooth inclined plane from rest is equal to that in which a body would fall through double the height of the plane; find the inclination.

18. A force equivalent to the weight of 9 lbs. pushes a mass of 6 lbs. up a smooth inclined plane of elevation 30°. Find the velocity of the mass in 3 secs., and the space described in that time.

19. A mass of 10 lbs. is pushed up a smooth inclined plane of length 50 feet and rising 3 in 200 by a uniform force in 5 seconds. Find the magnitude of the force.

20. How far will a force of 1000 lbs. wt. push a train of 100 tons mass down a smooth inclined plane, falling 1 in 200, in 7 minutes?

21. What force would be necessary to pull a train of 160 tons mass up a smooth inclined plane, rising 1 in 160, with an acceleration of 2 ft.-sec. units?

22. A train of 100 tons mass is pulled up an inclined plane supposed smooth and rising 1 in 200 at a uniform speed. Find the pulling force. What would be the force if the train were held at rest on the inclined plane? Give a reason for your answer.

23. A man descends a toboggan slide 300 yards long and falling 1 in 9. Disregarding the resistance of the atmosphere and supposing the slide to be perfectly smooth, what is his velocity in miles per hour at the foot of the slide?

24. A train running at the rate of 30 miles an hour shuts off steam on reaching the foot of an incline rising 1 in 240. How far will it run up the incline?

25. Two smooth inclined planes of the same altitude, and of elevations α and β, stand back to back. A body projected up the first with a velocity u ascends it, and without losing any velocity at the turn descends the second plane. Find its velocity at the foot of the second plane.

26. A body of mass m lbs. on an inclined plane of elevation α is acted upon by a horizontal force of P poundals, and moves up the plane. Find its acceleration.

27. A truck of 10 tons mass is pushed down an incline falling 1 in 200 by a steady force of 8 lbs. wt. If the incline is 140 yds. long find the velocity of the truck at its foot.

[In some of the following examples it will be necessary to use Tables of sines, cosines, etc.]

28. A body is projected up an inclined plane of elevation 20° with a velocity of 48 ft. per sec. and just reaches the top. Find the height of the plane.

29. A mass of 4 lbs. is pushed up an inclined plane of elevation 40° by a steady force of 3 lbs. wt. acting along the plane. Find its acceleration in ft.-sec. units correct to two decimal places.

30. A train weighing 200 tons running down an incline of 1 in 100 with a velocity of 30 miles an hour is brought to rest in half a mile by the steady action of its brakes. Find, to the nearest ton weight, the force exerted by the brakes.

31. Two particles start simultaneously from rest, the one down an inclined plane AC of length 25 feet, the other down a plane BC of length 70 ft., the heights of A, B above the horizontal plane through C being 7 and 56 ft. respectively. Find which particle will arrive at C first (the planes being smooth) and when at C how far it will be from the other particle.

32. Find to the nearest tenth of a lb. wt. the force which will push a mass of 4 lbs. up an inclined plane of elevation 35° with an acceleration of 3 ft.-sec. units.

33. In moving up a smooth inclined plane of elevation 42° over a space of 51 feet, the velocity of a mass of 3 lbs. increases from 4 to 13 ft. per sec. Find to the nearest lb. wt. the value of the force pushing it up the plane.

34. One particle slides from rest down a smooth inclined plane of elevation 30°. Two seconds after it has started, a second particle is projected down the plane with a velocity of 48 ft. per second. When and where will it overtake the first?

35. A body is projected down a smooth inclined plane and is observed to pass over 5 ft. in the first second, and 17 ft. in the next second. Find the inclination of the plane.

36. A mass of 5 lbs. is attached by a hook to a spring balance. The balance and the body are pulled up an inclined plane of elevation 30° with an acceleration of 4 ft.-sec. units. What weight is indicated by the balance during the motion?

37. A body is projected up a smooth inclined plane along a line of greatest slope with a given velocity. Prove that on returning to its starting point it has the same velocity as at the start.

38. A body is projected up a smooth inclined plane along a line of greatest slope with a given velocity. Prove that its time of ascent is equal to its time of descent to its starting point.

39. Divide a given smooth inclined plane of length 288 feet and height 64 feet into three parts, so that a particle sliding down it from rest may describe the three parts in equal times. In what time will each part be described?

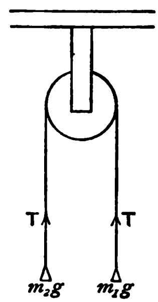
40. If a ball rolls without friction down an inclined plane, and in the 5th second after starting passes over 2207·25 centimetres, find the inclination of the plane to the horizon.

CHAPTER VII.

MASSES CONNECTED BY STRINGS PASSING OVER PULLEYS.

59. *Two unequal masses are connected by a light inextensible string passing over a smooth pulley; to find the acceleration of the system, and the tension of the string.*

Let m_1 and m_2 be the masses, m_1 being the heavier. Since the pulley is smooth, the tension of the string is the same throughout; let it be T poundals.



Since the string is inextensible, the one body will move upwards as fast as the other moves downwards, i.e. the velocities of the bodies at any instant are the same, and hence their accelerations are also the same. Let f be this common acceleration.

Considering the descending body: it is acted on by

(1) a force of T poundals vertically upwards,

(2) its own wt. m_1g .... downwards.

$$\therefore \; m_1g - T = m_1f \text{............(1)} \quad (P = mf).$$

Considering the rising body: in the same way

$$T - m_2g = m_2f \text{..............(2)}.$$

Adding (1) and (2)

$$(m_1 - m_2)\,g = (m_1 + m_2)\,f,$$

$$\therefore \; f = \frac{m_1 - m_2}{m_1 + m_2} \cdot g.$$

Multiplying across in (1) and (2), and dividing through by f,

$$m_1 m_2 g - m_2 T = m_1 T - m_1 m_2 g,$$

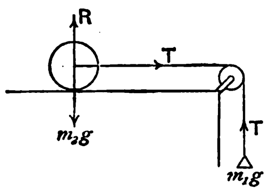
$$\therefore\ T = \frac{2m_1 m_2 g}{m_1 + m_2} \text{ poundals} = \frac{2m_1 m_2}{m_1 + m_2} \text{ lbs. wt}$$

N.B. The pressure on the pulley $= 2T$ poundals

$$= \frac{4m_1 m_2}{m_1 + m_2} \text{ lbs. wt.}$$

60. *One mass hanging freely draws another mass along a smooth horizontal plane by means of a light inextensible string passing over a smooth pulley at its edge: find the acceleration of the system and the tension of the string.*

Let m_1, m_2 be the masses of the bodies, T poundals the tension of the string, R poundals the normal reaction of the table, f the common acceleration.



The weight $m_2 g$ upon the table is balanced by R, and will not affect the horizontal motion of the body.

Considering the hanging body:

$$m_1 g - T = m_1 f \ldots\ldots\ldots\ldots(1) \qquad (P = mf)$$

Considering the body on the table:

$$T = m_2 f \ldots\ldots\ldots\ldots\ldots(2) \qquad (\ \text{,,} \quad \text{,,}\ )$$

Adding (1) and (2)

$$m_1 g = (m_1 + m_2) f,$$

$$\therefore\ f = \frac{m_1 g}{m_1 + m_2}.$$

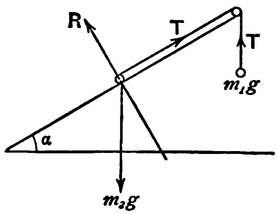
Multiplying across in (1) and (2), and dividing through by f,

$$m_1 m_2 g - m_2 T = m_1 T,$$

$$\therefore\ T = \frac{m_1 m_2 g}{m_1 + m_2}.$$

61. *One mass hanging freely draws another mass up a smooth inclined plane by means of a light inextensible string passing over a smooth pulley at the top of the plane: find the acceleration of the system, and the tension of the string.*

Let m_1, m_2 be the masses, T poundals the tension of the string, R poundals the normal reaction of the plane, f the common acceleration, α the elevation of the plane. The body on the plane has no motion at right angles to the plane, therefore resolving in that direction



$$R - m_2 g \cos\alpha = 0. \quad (P = mf)$$

Considering the body on the plane:

$$T - m_2 g \sin\alpha = m_2 f \text{.........(1)} \quad (P = mf)$$

Considering the hanging body:

$$m_1 g - T = m_1 f \text{...........(2)} \quad (\ ,,\quad ,,\ )$$

Adding (1) and (2)

$$m_1 g - m_2 g \sin\alpha = (m_1 + m_2) f,$$

$$\therefore\ f = \frac{m_1 - m_2 \sin\alpha}{m_1 + m_2} \cdot g.$$

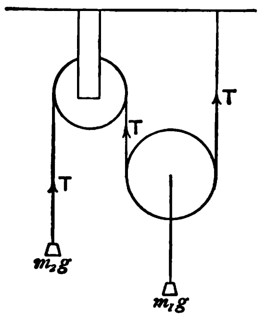
Multiplying across in (1) and (2), and dividing through by f,

$$m_1 T - m_1 m_2 g \sin\alpha = m_1 m_2 g - m_2 T,$$

$$\therefore\ T = \frac{m_1 m_2 g (1 + \sin\alpha)}{m_1 + m_2}.$$

62. *One end of a string is fixed; it then passes under a moveable pulley to which a mass m_1 is attached. The string then passes over a fixed pulley, and a mass m_2 is attached to its other end, all three sections of the string being vertical. Neglecting the masses of the pulleys, find the acceleration with which m_1 ascends, and the tension of the string.*

When the mass m_1 rises through one foot, *each* of the strings on either side of the moveable pulley must be shortened one foot, therefore 2 feet of string will slide over the fixed pulley, i.e. m_2 will fall through 2 feet.



Hence if f be the acceleration of m_1 upwards, $2f$ will be the acceleration of m_2 downwards.

Considering the mass m_2:

$$m_2g - T = m_2 . 2f ......(1). \qquad (P = mf)$$

Considering the mass m_1:

$$2T - m_1g = m_1f ......(2). \qquad (P = mf)$$

Therefore multiplying equation (1) by 2, and adding,

$$(2m_2 - m_1)g = (4m_2 + m_1)f,$$

i.e.

$$f = \left(\frac{2m_2 - m_1}{4m_2 + m_1}\right)g.$$

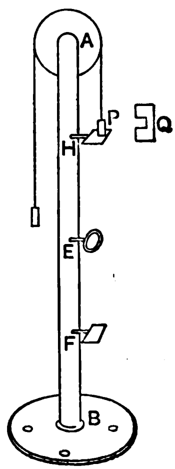
Again, multiplying across in equations (1) and (2),

$$m_1f(m_2g - T) = 2m_2f(2T - m_1g).$$

$$\therefore \; T(m_1 + 4m_2) = 3m_1 m_2 g,$$

i.e. $$T = \frac{3m_1 m_2 g}{m_1 + 4m_2} \text{ poundals.}$$

63. **Atwood's Machine** consists of a firmly fixed vertical pillar AB, carrying at the top a light pulley made to run as smoothly as possible. Over this pulley runs a thin inextensible string with equal masses (P) attached to its ends. Q is a small detached mass, called a *rider*, of such a shape that it may be placed on one of the masses P, but that it will not pass through the ring E, which can be fixed at any point of the column. F and H are small platforms which can also be attached to the column at any point.



In experiments with this machine, the rider Q is placed upon P resting on the platform at H: the platform is removed and the masses P and Q therefore descend together with acceleration

$$\frac{Qg}{2P + Q}. \text{ (Art 59.)}$$

When P passes through the ring at E, Q is removed and therefore from this point there is no moving force acting upon the system, and therefore no acceleration, i.e. the masses (P) move with uniform velocity.

To secure as great accuracy as possible :—

(1) The axis of the pulley is made to run upon the circumferences of other small wheels (called *friction wheels*). This considerably reduces the friction.

(2) The column AB is graduated accurately.

(3) A clock beating exact seconds is provided.

(4) A piece of apparatus is attached to the column so as to be able to remove the platform at H at any required

instant. No great degree of accuracy however can be obtained, for we have to neglect the masses of the pulley and string, and to neglect friction.

64. *Experiment* i. *To determine the value of g by a single experiment.*

We take a clock or metronome beating exact seconds. The mass Q is placed upon P at H, and the platform at H is dropped at a given second. Motion ensues. The platform F is so placed that the mass $(P+Q)$ strikes it after exactly t seconds, the ring at E being removed, and the distance HF is measured.

Let $HF = h$, $f =$ the acceleration,

then $$h = \tfrac{1}{2}ft^2, \qquad (s = ut + \tfrac{1}{2}ft^2)$$

$$f = \frac{Qg}{2P+Q}; \qquad (P = mf)$$

$$\therefore \frac{2h}{t^2} = \frac{Qg}{2P+Q},$$

i.e. $$g = \frac{(2P+Q)\,2h}{Qt^2}.$$

Since we know the values of P, Q, h, t, this equation gives us the value of g.

Experiment ii. *To verify the first law of motion.*

The experiment is conducted as before, but the ring E is so placed that the mass Q strikes it at a particular second and is left behind. The platform F is placed so that the mass P strikes it t seconds afterwards.

The distance EF is now measured, and by giving different values to t we can shew that the distance traversed after Q has been removed is proportional to the time, i.e. that the velocity is constant.

Experiment iii. *To verify the second law of motion.*

Proceeding as in Experiment ii., arrange the platform H and the ring E so that the mass Q remains on P for one

second; then observe the time (t) P takes to move from E to F.

Then $EF = vt$, where v is the velocity with which P moves from E to F, i.e. we can determine the velocity (v) generated by the force Qg in unit time.

(1) By varying the mass Q, but keeping the mass moved $(2P + Q)$ constant, we can shew that the velocity generated in unit time, i.e. the acceleration, varies as the moving force.

(2) Also by keeping Q constant, and varying P, we can prove that the acceleration varies inversely as the mass moved $(2P + Q)$. Hence, by the theory of algebraic variation, when both the moving force and the mass moved vary, the acceleration will vary directly as the moving force and inversely as the mass moved; i.e.

$$\text{acceleration} \propto \frac{\text{moving force}}{\text{mass moved}}.$$

EXAMPLES. VII.

[*In all the following examples the pulleys may be taken as smooth and of negligible mass, and the strings as weightless.*]

1. Two bodies of masses 5 lbs. and 3 lbs. are connected by a string passing over a pulley. Find the acceleration of the system, the tension of the string, and the space described by each body in 2 seconds from rest.

2. Two masses connected by a string hanging over a pulley move over 18 feet in 3 seconds from rest. If the heavier be a mass of 6 lbs., find the mass of the other.

3. Two masses, each equal to m, are connected by a string passing over a pulley. A third mass m is added to one of them; find their velocity when they have passed over 27 feet from rest.

4. Masses of 7 ozs. and 9 ozs. are connected by a string passing over a peg. Find the pressure on the peg.

5. If two masses, each equal to 3 lbs., connected by a string hang over a pulley and 6 lbs. be added to one of them, by how much is the pressure on the peg increased?

6. Two masses of 14 lbs. and 18 lbs. connected by a string hang over a pulley. After 4 seconds the heavier strikes the ground; how long after this will the other come to rest?

7. Two unequal masses connected by a string hang over a pulley. When they have attained a velocity v, the descending body is suddenly stopped, and instantly let go again. Find the time that elapses before the string again becomes tight.

8. Two masses of 4 lbs. and 6 lbs. connected by a string hang over a pulley, and after 5 seconds the string is cut: find the space described by each body in the next second.

9. In what time will a mass of 2 lbs. hanging freely draw a mass of 3 lbs. through a distance of 10 feet on a smooth horizontal table?

10. A mass of 2 lbs. hanging freely draws a mass of 4 lbs. along a smooth horizontal table: find the tension of the connecting string.

11. A mass of 10 lbs. lying on a horizontal table 24 feet from its edge is drawn along the table by a mass of 2 lbs. hanging freely: how long does it take to reach the edge?

12. A mass m hanging freely draws an equal mass m up an inclined plane of elevation 30°: find its acceleration, and the tension of the connecting string.

13. A mass of 10 lbs. is drawn up an inclined plane rising 3 in 5 with an acceleration $\frac{3g}{11}$ by a freely hanging body of mass m. Find the value of m.

14. A mass of 10 lbs. hanging freely draws an equal mass up an inclined plane through a space of 27 feet in 3 seconds. Find the elevation of the plane.

15. A string has masses of 9, 2, 5 lbs. attached to it at different points, in this order. The string passes over a smooth pulley so that the 2 and 5 lbs. masses are on one side, the 9 lbs. on the other: find the tensions of the two portions of string.

16. Two masses each of 2 lbs. hang over a pulley; a third mass of 2 lbs. is laid upon one of them for three seconds and then removed: how far will the system travel in the next 4 seconds?

17. Masses of 5 lbs. and 7 lbs. are fastened by separate strings to a mass of 8 lbs., and both strings pass over a smooth pulley: find the tensions of the strings.

18. Two masses m_1, m_2 hang over a pulley: find the ratio of m_1 to m_2 if they describe 36 feet in 3 seconds from rest.

*19. Prove that when two unequal masses hang over a smooth peg, the pressure on the peg is less than the sum of the weights.

20. Two equal masses hang at rest over a smooth pulley; one is projected upwards with a velocity of 64 feet per second: in what time will the string become taut again?

21. Masses of 5 and 3 lbs. rest on two inclined planes, each of elevation 30°, and are connected by a string passing over the common vertex: find the acceleration and the tension of the string.

22. Equal masses rest on two inclined planes of elevation 30° and 60° respectively, and are connected by a string passing over the common vertex; find the space described by each in 2 seconds.

23. Masses of 3 and 8 lbs. are connected with a mass of 5 lbs. by a string over a pulley. After one second the 8 lb. mass is removed: how long after this will it be before the system comes instantaneously to rest?

24. A mass $2m$ sliding on an inclined plane of height $48(\sqrt{3}-1)$ feet, and elevation 60°, draws a mass m vertically upwards from the ground to the top of the plane by means of a string passing over the top of the plane. How long does it take to do this?

25. A light string carrying two unequal weights and passing over a smooth fixed pulley can only just bear a tension equal to $\frac{4}{9}$ of the sum of the weights: prove that the least acceleration possible is $\frac{g}{3}$.

26. In an Atwood's machine the equal masses are each 3 ozs., and when an extra mass of 2 ozs. is placed upon one of them, it is noticed that it descends through 16 feet in 2 seconds. Hence shew that $g=32$.

27. The equal masses in an Atwood's machine are each 979 grammes, and the rider 4 grammes. When the rider is attached the descending body is observed to pass through one metre in 10 seconds: hence shew that $g=981$.

28. Two bodies, each 1000 grammes in mass, are connected by a string which passes over a frictionless pulley; a rider of 15 grammes weight is placed on one of the bodies and the system moves from rest through 200 cms. in 7·4 seconds: find the acceleration produced, and calculate a value for the acceleration due to gravity.

*29. Masses nP and P are connected by a string which passes over a smooth pulley, and at the end of each second from the beginning of the motion a mass P is taken from the first and added to the second; shew that after n seconds the motion will be reversed.

*30. A heavy particle is resting on an inclined plane, inclined at an angle a to the horizon, and is held up by a string which passes over the summit of the plane (supposed smooth) and is attached to an equal particle resting on a smooth horizontal plane, the angle of inclination to the horizon of both parts of the string being a. Shew that the particle on the horizontal plane begins to move along that plane with an acceleration

$$g\,\frac{\sin 2a}{3+\cos 2a}.$$

31. Two unequal masses connected by a string are placed one on a smooth horizontal plane and the other on a smooth inclined plane, their common edge being perpendicular to the vertical plane in which the masses move; find the acceleration of each mass and the tension of the string.

32. A fixed pulley lies on a smooth inclined plane, elevation a; the ends of the string passing over the pulley are attached to weights W_1 and W_2, of which W_1 lies on the plane, and W_2 is suspended by the string over the lower edge of the plane under the action of gravity: find the acceleration of W_1, the motion of each weight being rectilinear.

33. Deduce the value of g from the following experiment with an Atwood's machine: the weights $=16\frac{1}{8}$ ozs. and $15\frac{7}{8}$ ozs. respectively; time of falling 8 ft. $=8$ seconds.

34. Masses of 3 lbs. and 5 lbs. are attached to the ends of a string which passes over a light frictionless pulley. If the string breaks after the system has been moving for 4 secs. from rest, find the greatest height above its starting point which the 3 lb. mass will reach.

*35. Two equal masses attached by an inextensible weightless thread that passes over a light pulley hang in equilibrium. Shew that the tension of the thread is unaltered when $\frac{1}{n}$th of its mass is added to one, and $\frac{1}{n+2}$th of its mass is removed from the other.

36. Initially, while the string is tight, the upper and lower masses of an Atwood's machine are projected downwards and upwards respectively with such velocities that in $\frac{1}{8}$th second each moves through 49 cms. Find these initial velocities, and determine the ratio of the masses and the acceleration if after 3 more seconds the system comes to rest ($g=980$).

37. A double smooth inclined plane has each of its slant faces inclined to the horizon at an angle of 30°. If a mass m_1, lying on one face, is partially supported by two strings which have masses m_2, m_3 at their other ends and lying on the other face, and slides down; prove that the tension of each string is to the weight at its ascending end as

$$m_1 : m_1+m_2+m_3.$$

38. Masses of 3 lbs. and 4 lbs. are attached by separate strings to a mass of 5 lbs. and suspended over a pulley: find the tension of each string.

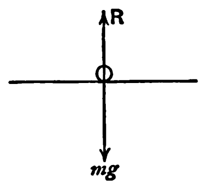
39. A string having masses of 7 lbs. and 5 lbs. attached to its ends, and a mass of 4 lbs. to a point in between, is hung over a pulley so that the 4 and 5 lbs. masses are on the opposite side of the pulley to the 7 lbs. mass. Find the tensions of the two parts of the string.

CHAPTER VIII.

BODIES ON MOVING HORIZONTAL PLANES, AND MOTION ON ROUGH PLANES.

65. **Pressure on a horizontal plane in motion.**

Let a body of mass m lbs. rest upon a horizontal plane which has an acceleration f, vertically upwards.



Let R be the pressure *on the plane*, vertically downwards.

By Newton's Third Law of Motion, the pressure of the plane on the body is equal and opposite to this.

Therefore there are two forces *acting on this body*,

(1) its weight mg poundals vertically downwards,

(2) the pressure R poundals vertically upwards.

By hypothesis their resultant, acting on mass m, produces an acceleration f vertically upwards.

$$\therefore\ R - mg = mf, \qquad\qquad (P = mf)$$

$$\therefore\ R = m\,(g + f) \text{ poundals.}$$

If the plane *descend* with acceleration f, we have, in the same way, $mg - R = mf$,

$$\therefore\ R = m\,(g - f) \text{ poundals.}$$

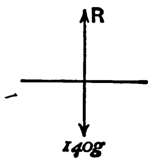
COR. If the plane move with uniform velocity $f=0$,

$$\therefore R=mg,$$

i.e. the pressure upon the plane is the same as if it were at rest.

66. Ex. i. *An ascending lift when near the top of the shaft is subject to a retardation $\frac{g}{10}$ ft.-sec. units; find the pressure on its floor during this time of retardation of a man who weighs* 10 *stone.*

Let R poundals be the pressure on the floor; then the resultant force *acting upon the man* is $(140g-R)$ poundals vertically downwards, causing a retardation $\frac{g}{10}$.

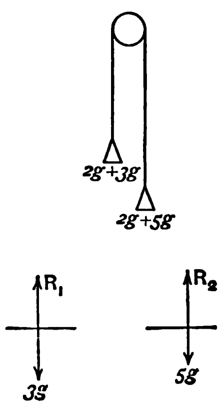


$$\therefore 140g-R=140\frac{g}{10}, \qquad (P=mf)$$

whence
$$R=140\left(1-\tfrac{1}{10}\right)g$$
$$=126g \text{ poundals}$$
$$=126 \text{ lbs. wt.}$$

N.B. The pressure on the floor of the lift is **independent of the velocity** of the lift.

Ex. ii. *Two scale-pans each of mass* 2 *lbs. connected by a string hang over a smooth pulley. A mass of* 3 *lbs. is placed in one and* 5 *lbs. in the other: find the pressures upon the pans during the ensuing motion.*



Let f be the acceleration of the system, R_1 poundals the pressure on the rising, R_2 poundals the pressure on the descending pan.

We find $\quad f=\dfrac{7g-5g}{12}$ (as in Art. 59) $=\dfrac{g}{6}$ ft.-sec. units.

Considering the mass 3 lbs. in the rising pan, the resultant force acting upon it is (R_1-3g) poundals vertically upwards;

$$\therefore\ R_1-3g=3f=\frac{3g}{6}=\frac{g}{2}, \qquad (P=mf)$$

$$\therefore\ R_1=\frac{7g}{2}\text{ poundals}$$

$$=3\tfrac{1}{2}\text{ lbs. wt.}$$

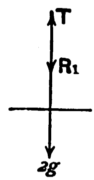
Considering the mass 5 lbs. in the descending pan, the resultant force acting upon it is $(5g-R_2)$ poundals vertically downwards;

$$\therefore\ 5g-R_2=5f=\frac{5g}{6}. \qquad (P=mf)$$

$$\therefore\ R_2=5g\left(1-\tfrac{1}{6}\right)$$

$$=\frac{25g}{6}\text{ poundals}$$

$$=4\tfrac{1}{6}\text{ lbs. wt.}$$



N.B. If we consider the motion *of the rising scale-pan*, we must remember that the resultant force acting upon it is $(T-R_1-2g)$ poundals vertically upwards, where T is the tension of the string.

EXAMPLES. VIII *a*.

1. A balloon ascends with an acceleration of 8 ft.-sec. units: find the pressure of a 56 lbs. weight on the floor of the car.

What would be the pressure if the balloon ascended with uniform velocity?

2. A bucket containing a cwt. of coal is being drawn up from a coal-pit, so that the pressure of the coal on the bottom of the bucket is equal to 126 lbs. weight: what is the acceleration of the bucket?

3. A mass of m lbs. is suspended by a string from the roof of a railway carriage moving with uniform velocity in a straight line: find the magnitude and direction of the tension of the string.

4. A mass of 40 lbs. is suspended by a rope from the car of a balloon which ascends with an acceleration of 4 ft.-sec. units: find the tension of the rope.

5. If a man holding a 14 lb. weight by means of a string jump off a height, what will be the tension of the string during his fall?

6. A body of mass 2 lbs. falling freely under the action of gravity has a velocity of 48 ft. per second: what force will bring it to rest in 3 seconds?

7. A body of mass 4 lbs. falls freely under the action of gravity for 3 seconds: what force will bring it to rest in the next 3 feet?

8. A cannon-ball of mass 10 lbs. after falling through 40 feet from rest penetrates 6 inches into a bank of mud: find the average pressure of the mud on the ball.

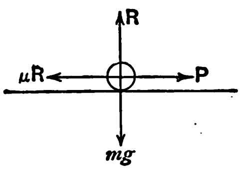
9. A man can just lift a mass of 20 stone off the ground. Shew that if he is in a cage descending with acceleration 4 ft.-sec. units, he can then lift 22 stone 12 lbs.

10. A body weighs, according to a spring balance at rest, 10 lbs. What weight will be indicated for the same body if the balance be in a balloon which is ascending with an acceleration of 16 ft.-sec. units?

11. Two scale-pans each of mass 4 ozs., connected by a string, hang over a pulley. A mass of 4 ozs. is placed in one of them; find its pressure on the pan during the ensuing motion.

67. *Acceleration on a rough horizontal plane.*

Let a horizontal force, P, act on a mass m resting on a horizontal plane whose coefficient of friction is μ. It is required to find f the acceleration produced.



Let R be the normal reaction of the plane; then μR is the force of friction which will act in a direction opposite to that of P.

The body moves along the plane, i.e. there is no motion in a vertical direction,

$$\therefore\ R = mg \qquad\qquad (1).$$

Also the resultant force in the direction of $P = P - \mu R$,

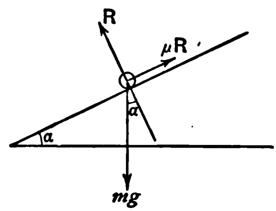
$$\therefore\ P - \mu R = mf. \qquad\qquad (P = mf)$$

Therefore from (1)

$$f = \frac{P - \mu mg}{m}.$$

68. *Acceleration of a particle sliding down a rough inclined plane.*

Let m be the mass of the particle, R the normal reaction of the plane, μ the coefficient of friction, α the elevation of the plane, f the acceleration down the plane. Then μR is the force of friction acting up the plane.



The particle moves on the plane, i.e. there is no motion in the direction at right angles to the plane.

$$\therefore\ R - mg\cos\alpha = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The resultant force acting on the particle down the plane is

$$\begin{aligned} &mg\sin\alpha - \mu R \\ &= mg\sin\alpha - \mu mg\cos\alpha, \text{ from (1)}, \\ &= mg(\sin\alpha - \mu\cos\alpha); \\ \therefore\ &mg(\sin\alpha - \mu\cos\alpha) = mf, \qquad (P = mf) \end{aligned}$$

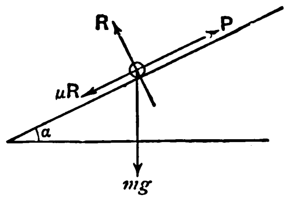
and $$f = g(\sin\alpha - \mu\cos\alpha).$$

COR. If the particle be projected *up* the plane, the frictional force will act down the plane, and we shall find that the acceleration is then $g(\sin\alpha + \mu\cos\alpha)$ down the plane.

69. *Acceleration of a particle under the action of a constant force up an inclined plane.*

Let the force P, acting on the mass m, draw it up the inclined plane of elevation α. Also let R be the reaction of

the plane, μ the coefficient of friction, f the acceleration up the plane.



The body moves on the plane, i.e. there is no motion at right angles to the plane.

$$\therefore\ R - mg\cos\alpha = 0 \ldots\ldots (1).$$

The resultant force up the plane is

$$P - mg\sin\alpha - \mu R$$
$$= P - mg\sin\alpha - \mu mg\cos\alpha, \text{ from (1).}$$
$$\therefore\ P - mg(\sin\alpha + \mu\cos\alpha) = mf,$$

i.e.
$$f = \frac{P - mg(\sin\alpha + \mu\cos\alpha)}{m}.$$

EXAMPLES. VIII *b*.

1. A mass of 3 lbs. hanging freely draws an equal mass, by means of a string, along a rough horizontal table. Find the space described by either body in one sec. if the coefficient of friction is $\frac{1}{3}$.

2. A ball is projected along a rough horizontal plane with a velocity of 24 feet per sec.: how far will it go before it is reduced to rest, when the coefficient of friction is $\frac{1}{3}$?

3. What force must be exerted by an engine to move a train of 120 tons with an acceleration of 10 ft.-sec. units along a rough horizontal plane, if the resistance due to friction be 10 lbs. wt. per ton?

4. A particle of mass 10 lbs. moves along a horizontal plane against a friction of one-fifth of its weight for a distance of 20 feet before coming to rest: find its initial velocity.

5. A train of 250 tons weight, moving at 45 miles an hour, is stopped by the brakes in 220 yards. Find the value of the total frictional force in tons weight.

6. A body slides down a rough inclined plane: shew that its acceleration is $\frac{g\sin(\alpha-\phi)}{\cos\phi}$, where ϕ is the angle of friction.

7. A train ascends a gradient of 1 in 40 by its own momentum for a distance of one mile, the resistance from friction etc. being 10 lbs. per ton: find its initial velocity in miles per hour.

8. The weight of a train is 150 tons, and it is kept going with uniform velocity by the pull of the engine, the resistance due to friction being 30 lbs. weight per ton. Find the acceleration of the train when a slip-carriage weighing 30 tons is detached, if the engine continues to pull with the same force.

9. A weight W is placed on a rough horizontal table, and is moved along the table by a weight, P, which hangs over the edge of the table and is attached to W by an inextensible string: shew that the acceleration of W is $\frac{P-\mu W}{P+W}g$, where μ is the coefficient of friction.

10. An engine exerting a force equal to 3 tons weight pulls a train of mass 100 tons up an incline of 1 in 100: taking the frictional resistance as 11·2 lbs. weight per ton, find the acceleration of the train.

11. A mass of 4 lbs. hanging freely over the edge of a rough table pulls, by means of a string, a mass of 2 lbs. along the table through a distance of 2 feet in half a second: find the coefficient of friction of the table.

12. A body in sliding down a rough inclined plane, of height h and elevation a, acquires a velocity v: find the coefficient of friction.

13. A body just rests on a plane whose inclination is 30°; find the velocity acquired by a body in sliding from rest down the length of the same plane when it is lifted to an elevation of 60°, and its vertical height is 150 feet. (Assume that the statical coefficient of friction is equal to the dynamical.)

14. Two planes of elevation 30° and 60° respectively and of equal heights are placed back to back: a mass $2m$ is placed on the steeper plane and connected by a string with a mass m on the other; if the steeper plane be smooth and the other rough $\left(\mu=\frac{1}{\sqrt{3}}\right)$ find the resulting acceleration.

15. A mass of 10 lbs. is sliding down a rough inclined plane $(\mu=\frac{1}{2})$ of elevation $\sin^{-1}\frac{3}{5}$, with a velocity of 30 feet a second: what force will stop it in 100 yards?

16. A body is projected up a rough inclined plane $\left(\mu=\frac{1}{2\sqrt{3}}\right)$ of elevation 30° with a velocity of 64 ft. per sec., and at the same instant another body slides from rest at the top. Find the time to their meeting point, if that point is half-way up the plane.

17. A mass of 5 lbs., drawn along a rough horizontal plane by a horizontal force of 6 poundals, acquires a velocity of 2 feet per sec. in moving through 2 feet from rest. Compare the force due to the friction of the plane with the weight of the body.

18. A body falls down a rough inclined plane of length l in the time T. If it be projected up the plane with the same velocity with which it reached the ground, find the values of $\frac{l'}{l}$ and $\frac{T'}{T}$, l' being the portion of l which it ascends, and T' the time of describing l', μ being the coefficient of friction, a the slope.

19. If a body projected upwards along a rough inclined plane of elevation 30°, ascends during 3 seconds, and then slides down to the point of projection in $3\frac{1}{2}$ seconds; find the ratio of the weight of the body to the force which the friction of the plane exerts upon it during the motion.

CHAPTER IX.

REVISION QUESTIONS (ORAL), PAPERS AND MISCELLANEOUS EXAMPLES.

REVISION QUESTIONS. IX *a*. (May be taken Orally.)

1. A man in a train between two stopping stations 5 miles apart remarked, "We are travelling at the rate of 40 miles an hour." Explain clearly what he meant.

2. If you wish to find the *average velocity* of a train between two stations, what data must you have?

3. Define acceleration. What is the meaning of the word 'rate' in the definition?

4. A man drops a sand-bag from a rising balloon. What is the point to notice if you have to investigate the motion of the sand-bag?

5. Cut a piece of paper to a size somewhat smaller than a penny. Placing the paper on the penny and holding it in a horizontal position, let both fall. The paper and the penny will reach the ground together. What can you deduce?

6. At the end of consecutive seconds the velocity of a body is 5, 7, 9, 11, 13... ft. per second. What can you infer as to its motion?

7. How would you prove experimentally that the weight of a body is not a constant quantity?

8. How would you show experimentally that the weights of two bodies, *at the same place*, are proportional to their masses?

9. What important assumption is made in obtaining the formula

$$P=mf?$$

10. "The British absolute unit of force." What is it? What is the meaning of the word *absolute*?

11. Is a pound-weight an absolute force?

12. At the end of consecutive units of time the momentum of a body is found to be $11m$, $9m$, $7m$, $5m$ and so on, where m is the mass of the body. What can you infer?

13. Two masses are connected by a light inelastic string. The string being held taut initially, the bodies slide down an inclined plane along a line of greatest slope. What will be the tension of the string during the motion?

What difference would it make if the bodies did not slide down the same line of greatest slope?

14. A point has three velocities of 6, 6, and 9 ft. per sec. making angles of 120° with one another. What is its resultant velocity?

15. A point has two equal velocities of 3 miles an hour making an angle of 120° with one another. What is its resultant velocity?

[*Fill in the blank spaces in Questions* 16—19.]

16. A velocity of 2 ft. per sec.

= a velocity of ft. per minute,
= „ „ ft. per hour,
= „ „ yds. per sec.
= „ „ yds. per minute.

17. An acceleration of 3 ft.-sec. units

= an increase of velocity of 3 ft. per sec. during every sec.
= „ „ ft. per sec. „ minute,
= „ „ ft. per minute „ „
= „ „ ft.-minute units.

18. An acceleration of 30 mile hour units

= an increase of miles per hour during every hour,
= „ „ „ „ „ minute,
= „ „ „ minute „ „
= „ „ feet „ „ „ „
= ft.-minute units.

19. The acceleration of gravity

= an increase of velocity of 32 ft. per sec. during every sec.
= „ „ „ ft. per minute „ „
= „ „ „ ft. per minute „ minute.
= ft.-minute units.

∴ the measure of g in ft.-minute units is .

20. A body subject to an acceleration of 8 ft.-sec. units starts from rest. What is its velocity after one second, after two seconds, after 10 seconds, after t seconds of motion?

21. A body subject to an acceleration of 6 ft.-sec. units starts with a velocity of 2 ft. per sec. What is its velocity after one second, after two seconds, after three seconds, after t seconds of motion?

22. It would be just as easy to move a 100 ton weight along a smooth horizontal table as a 100 lb. wt. How is this? Would the accelerations of the two bodies be the same if equal forces were applied?

23. A particle travels with uniform velocity along the circumference of a circle. Some force must be acting upon the particle. Why? Also this force must act towards the centre of the circle. Why?

24. A heavy particle is attached to a fixed point on a smooth horizontal table by a string. The string being held taut the particle is started at right angles to the string with a velocity v along the table. It revolves round the fixed point with the constant velocity v. Why?

25. Would it be possible to walk along a smooth horizontal surface?

REVISION PAPER. IX b.

1. A point travels over a miles in b hours. Express its average velocity in feet per second.

2. A particle travelling with a velocity of 42 yards per second is brought to rest in 14 secs. by a uniform retardation: find the value of the retardation in ft.-sec. units.

3. A ball thrown vertically upwards with a velocity of 128 ft. per sec. passes through a sheet of glass after 2 secs. and so loses half its velocity. What interval will elapse between its striking the glass and reaching its highest point?

4. A mass of 120 lbs. is pushed up an inclined plane rising 11 in 120 with an acceleration of 4 ft.-sec. units: find the moving force.

5. Two masses of 4 lbs. and 8 lbs., connected by a light string, hang over a smooth pulley and move freely from a position of rest. Find the acceleration of each mass, the tension of the string, the velocity of each mass after 3 seconds of motion, and the further height to which the 4 lb. mass would rise if the heavier body were then stopped.

6. The velocity, v ft. per sec., at time t of a moving particle is given by the following table:—

$t=$	0	1	2	3	4	5	...
$v=$	11	9	7	5	3	1	...

Plot the corresponding values of v and t. Find out all you can about the motion of the particle, and the time when it comes to rest.

REVISION PAPER. IX *c*.

1. A point on the equator moves through 25000 miles in 24 hours. Find its speed in feet per second to the nearest integer.

2. A stone drops from rest from a height of 160 feet. Neglecting atmospheric resistances, find how long it takes to reach the ground, and calculate its velocity just before it touches the ground. Plot a curve roughly showing the velocity of the stone at various heights.

3. A man of weight 150 lbs. stands on a platform; what pressure does he exert on it if the platform moves vertically with an acceleration of 2 ft. per sec. per sec. (1) upwards, (2) downwards?

4. A particle under a uniform acceleration passes over $19\frac{1}{2}$ ft. in the fifth second, $28\frac{1}{2}$ ft. in the eighth second of its motion. Find the acceleration and the initial velocity.

5. Two smooth planes of equal altitudes stand back to back. A mass of 8 lbs., connected by a light string passing over the top of the planes with a mass of 5 lbs., pulls it up its plane with an acceleration of 12 ft.-sec. units. If 30° is the elevation of the plane on which the 5 lb. mass slides, find the elevation of the other.

6. If the unit of force acting on the unit of mass produced two units of acceleration, the formula $P=mf$ would not be true. What would it become?

REVISION PAPER. IX *d*.

1. A point travels for 4 secs. at the rate of 3 ft. per sec., for 2 secs. at the rate of 5 ft. per sec., and for 3 secs. at the rate of 2 ft. per sec. What is its average velocity during this motion?

2. *ABC* is a straight line, the points *A* and *B* being 20 feet apart. One particle is projected along *AC* from *A* with an initial velocity of 37 ft. per sec., and an acceleration of 4 ft.-sec. units. At the same instant, a second particle starts from *B* and moves along *BC* with a uniform velocity of 40 ft. per sec. Find when and where the one particle catches the other up.

3. A point starting with a velocity of 3 ft. per sec. has a uniform acceleration of ·25 ft.-sec. units in the direction of its initial velocity. Draw a graph of its velocity-time equation, and *from the graph* determine the space describe in 8 seconds.

4. A man points his boat, and rows it with a velocity of 8 miles an hour, straight across a stream a quarter of a mile wide. If he lands 330 yards below the point opposite his starting place, find the velocity of the current.

5. Two bodies of mass 1 and 3 lbs. are connected by a light string passing over a smooth pulley. If the string is cut 4 secs. from the beginning of motion, find how much further the lighter body will ascend.

6. A light string passing over a smooth pulley has a spring balance weighing 6 ozs. attached to one end, and a weight of 16 ozs. to the other. A weight of 8 oz. is placed in the balance, and the system is allowed to move freely. What weight is indicated by the balance?

REVISION PAPER. IX *e*.

1. A train travels uniformly over 1320 yards in a minute: find its velocity in miles per hour.

2. If a ball is thrown vertically upwards with a velocity of 234 ft. per sec., after what time will its velocity be 102 ft. per sec. downwards, neglecting atmospheric resistance?

3. A mass of 500 pounds moving in a straight line is observed to change its speed from 60 miles an hour to 50 miles an hour in 10 seconds. What force must have acted to produce the change?

4. When a lift is at rest its floor is just strong enough to carry a load of one ton. If the lift is moving (1) with constant speed, (2) with an upward acceleration, (3) with a downward acceleration, can the floor support the same load?

5. A particle starting from rest moves under an acceleration of 4 ft.-sec. units. Find the corresponding values of s to the following values of v: 0, 2, 4, 6, 8. Plot the corresponding pairs of values, and read off, as accurately as you can, the velocity of the particle when it has passed over $3\frac{1}{2}$ feet, and the space passed over when its velocity is 7 ft. per sec.

6. Given $g = 981$ cm.-sec. units, find its value in ft.-sec. units. [A metre = 39·37 inches approx.]

MISCELLANEOUS EXAMPLES. IX *f*.

1. A particle subject to a uniform acceleration describes 50 feet and 66 feet in the 3rd and 5th seconds of its motion respectively: find its initial velocity.

2. A body falls through a height of 16 feet and then striking a pane of glass loses half its velocity. It then reaches the ground with a velocity of 80 ft. per sec.; find the height of the glass above the ground.

3. A ball is let fall from the top of a tower 384 ft. high; 2 secs. later another is projected vertically upwards from the foot of the tower with a velocity of 128 ft. per sec.: when and where do they meet?

4. A body starting from rest falls in the last second of its motion 19 times as far as in the 1st second: find the whole space described.

5. A body, sliding from rest down a smooth inclined plane, passes over 1040 ft. in 13 secs.: find the ratio of the height of the plane to its length.

6. AB is the vertical diameter of a circle in a vertical plane; if a body in sliding down a chord AP acquires one-half the velocity it would acquire in falling from A to B, find the inclination of AP to the vertical.

7. A man in a lift rising with a velocity of 8 ft. per sec. drops a ball which reaches the ground in 5 secs.; find the height from which the ball was dropped.

8. During how many minutes must a force equal to the weight of 1 ton act upon a train weighing 150 tons to generate in it a velocity of 30 miles an hour?

9. The mass of a train is 200 tons, and the resistance due to friction etc. is 8 lbs. wt. per ton. If the tractive force upon it be equal to the weight of 2 tons, find its acceleration.

10. If the depth of a well be 144 ft. and sound travel with a velocity of 1120 ft. per sec., find the time that elapses after dropping a stone before hearing it strike the water.

11. Find the statical measure of a force which in half a mile will stop a train of 100 tons, moving at the rate of 30 miles an hour on smooth horizontal rails.

12. If a train weighing 200 tons, and moving at the rate of 30 miles an hour, can be stopped by the action of the brakes within a space of 60 yards, find the amount of the friction in tons weight.

13. A mass m_1 hanging freely draws a mass m_2 up an inclined plane of elevation 45° by means of a smooth inextensible string over the top of the plane: if they move with acceleration $\frac{g}{4}$ find the ratio of m_1 to m_2.

14. A mass m is pushed up a smooth inclined plane of length l and height h by a uniform force in t seconds: find the magnitude of the force.

15. A mass of 4 ozs. is connected by means of a string over a smooth pulley with a mass m; when m has been in motion for 3 secs. the string is suddenly cut; find m so that the 4 oz. mass may ascend $\frac{16}{9}$ths feet further before it begins to descend.

16. What must be the unit of space, if gravity be represented by the number 14, when the unit of time is 5 seconds? ($g = 32 \cdot 2$ ft.-sec. units.)

17. Two steamers, X and Y, are respectively at points A and B. X steams away with a uniform velocity of 10 miles an hour in a direction making an angle of 60° with AB. Find in what direction Y must start at the same moment if it steam with a uniform velocity of $10\sqrt{3}$ miles an hour, in order that it may come into collision with X.

18. A particle moves in a straight line along a smooth horizontal plane with a velocity of 6 ft. per second; after 2 secs. a velocity of 5 ft. per sec. is imparted to it in a direction at right angles to its original motion; find the distance of the particle from its starting point after it has been in motion for 4 seconds.

19. APB is a vertical circle whose highest and lowest points are A and B, and Q is taken in AB so that $AQ = AP$. If AP produced meet the tangent at B in R, and a body slide down APR from rest, prove that the times of the body being within and without the circle are in the ratio of AQ to BQ.

20. A body moving along a straight line is known to be acted upon by a constant force; at a certain instant it is moving at the rate of 12 feet a sec. and in the next 10 secs. it describes a distance of 470 feet: what velocity does it gain in each second of its motion?

21. A train moving uniformly describes 88 yds. in 3 secs.: find its rate in miles per hour. In what time will it travel 600 miles with a stoppage of 5 minutes after every 100 miles?

22. A body acted on by a constant force begins to move from a state of rest; it is observed to move through 55 feet in a certain 2 secs., and through 77 feet in the next 2 secs.: what distance did it describe in the first 6 secs. of its motion?

23. A particle moves under a
distances described in successive sec

24. Express an acceleration of 4 ft.-

25. The velocity of a point is incr
during every minute: find its accelerati

26. If two vertical circles touch e
and any straight line be drawn from th
meet the outer circle, shew that the t
from rest along the part of the line (c
intercepted between the circles is const

27. Three equal masses (m) are at
string, one at each end, the other in th
draw the third along a smooth horizon
the two portions of the string.

28. A train is moving on a horiz
weight of the train (exclusive of the e
resistance arising from friction etc. to b
of the couplings of the carriage whic
(1) when the velocity of the train is u
with an acceleration of 4 ft. per second,

29. The velocity of a ship in a st
$8\frac{1}{5}$ miles an hour; a ball is bowled acr
ship's length, with a uniform velocity o
the true path of the ball in space, an
45 ft. in 3 secs. nearly.

30. A body is subject to a unifor
the whole time of motion to be cons
described in the first m seconds, and
last m seconds, and s the total space de

31. If the unit of time be half a min
2 yds., what will be the measure of th
describes, at a uniform rate, 14 miles in

32. The side BC of a triangle ABC
times of falling down the two sides BA,
be isosceles or right-angled.

33. A flexible heavy string, length
fixed small pulley, the two unequal port
Prove that at the instant when its mid
below the pulley, the acceleration with

Find also the tension of the s
descending portion at the same insta

$\cdot OPF$ = a right angle, i.e. P is moving at right

...lar velocity about the point B

$$= \frac{2v\cos\frac{\theta}{2}}{BP} = \frac{2v\cos\frac{\theta}{2}}{2r\cos\frac{\theta}{2}} = \frac{v}{r},$$

...ngular velocity of P about the centre.

... the point P is moving at right angles to BP, and ...mes called the *instantaneous centre of rotation* of the

...ticle, at the point A, has a velocity u making an angle ... : to find the angular velocity of the particle at the point O.

...ngular velocity required.

... is equivalent to

$u\cos\theta$ along AO,

$u\sin\theta$ perpendicular to AO.

... $u\cos\theta$ has no angular velocity about O. $\therefore\ \omega$ = the ... of the component $u\sin\theta$ about O

$$= \frac{u\sin\theta}{AO}. \qquad \text{(Art. 92.)}$$

EXAMPLES. XI *d*.

Angular Velocity.

[Take $\pi = \frac{22}{7}$.]

...urns through three-quarters of a right angle in one ... angular velocity.

... describes a circle of radius 5 feet with the unit of ...; find its linear velocity.

... makes 50 revolutions per minute about its centre: ... velocity of any point on the wheel about its centre.

... moves on the circumference of a circle of radius 3 feet ... of 12 feet per sec.: find its angular velocity, and the ...s) its radius turns through in half a second.

... the angular velocities of the hour-hand, the minute- ...econd-hand of a watch.

... train moving with velocity v, a carriage, on a road ...line at a distance d from it, is observed to move so as to ... in a line with a more distant fixed object whose least ...he railway is D. Find the velocity of the carriage.

23. A particle moves under a constant acceleration: shew that the distances described in successive seconds are in arithmetical progression.

24. Express an acceleration of 4 ft.-sec. units in mile-minute units.

25. The velocity of a point is increased by 12,000 yards a minute during every minute: find its acceleration in ft.-sec. units.

26. If two vertical circles touch each other at their lowest point, and any straight line be drawn from that point to cut the inner and to meet the outer circle, shew that the time of a heavy particle falling from rest along the part of the line (considered as an inclined plane) intercepted between the circles is constant.

27. Three equal masses (m) are attached to a light inextensible string, one at each end, the other in the middle. Two hanging freely draw the third along a smooth horizontal table: find the tensions of the two portions of the string.

28. A train is moving on a horizontal railroad. Assuming the weight of the train (exclusive of the engine) to be 160 tons, and the resistance arising from friction etc. to be 8 lbs. per ton, find the tensions of the couplings of the carriage which is attached to the engine, (1) when the velocity of the train is uniform, (2) when it is moving with an acceleration of 4 ft. per second, per second.

29. The velocity of a ship in a straight course on an even keel is $8\frac{1}{5}$ miles an hour; a ball is bowled across the deck perpendicular to the ship's length, with a uniform velocity of 3 yards in a second; describe the true path of the ball in space, and shew that it will pass over 45 ft. in 3 secs. nearly.

30. A body is subject to a uniform acceleration. If t represent the whole time of motion to be considered, and if a be the space described in the first m seconds, and b the space described in the last m seconds, and s the total space described, shew that $s = \frac{(a+b)t}{2m}$.

31. If the unit of time be half a minute and the unit of length be 2 yds., what will be the measure of the velocity of a body which describes, at a uniform rate, 14 miles in 3 hours?

32. The side BC of a triangle ABC is vertical; shew that if the times of falling down the two sides BA, AC be equal, the triangle must be isosceles or right-angled.

33. A flexible heavy string, length $2l$, is moving over a smooth fixed small pulley, the two unequal portions of it hanging vertically. Prove that at the instant when its middle point is at a distance x below the pulley, the acceleration with which it is moving is $\frac{x}{l}g$.

Find also the tension of the string at any assigned point of the descending portion at the same instant.

34. If the unit of force were that which acting on 1 ton of mass would in 1 minute generate a velocity of 1 mile a minute, how many units of force would the weight of one ton contain?

35. Two particles are projected with a velocity of 40 ft. per sec. from points 88 ft. apart, the one up and the other down a rough plane ($\mu = \frac{1}{4}$) inclined to the horizon at an angle $\tan^{-1}(\frac{7}{24})$. Find when and where they will meet, and account for the double solution.

36. A person rows with a velocity of 6 miles an hour across a river a quarter of a mile wide, which runs with a velocity of 4 miles an hour. The head of the boat makes a constant angle θ with the bank while he rows across, and he arrives at a point 36 yds. 2 ft. lower down the bank than the point opposite his starting point. Prove that $\tan\theta = \frac{4}{3}$.

37. If the time of a body's fall from a certain height at one place on the earth's surface be m secs. less than at another place, and the velocity acquired in the fall be a ft. per sec. greater, prove that $\frac{a}{m}$ is the geometric mean of the accelerations of gravity at the two places.

38. If two weights be contained in scale-pans connected by a string over a smooth pulley, prove that if the weights of the pans be neglected the pressure between each pan and the contained weight is equal to the tension of the string.

39. Three equal weights are fastened to a string whose length (l) is equal to that of a smooth inclined plane; one weight is attached to each end, and the other weight to the middle of the string; when one weight hangs over the top of the plane the weights are in equilibrium; if the second weight also is just made to hang vertically, find the velocity with which the third weight reaches the top of the plane.

40. A railway train, running at the rate of 40 miles an hour, is detached from the engine at the foot of a smooth incline of 1 in 50: how long and how far will it move up the plane?

41. A ship is sailing at the rate of $10\frac{10}{11}$ miles an hour; across the deck, perpendicular to the direction of the ship's course, a ball is bowled with a uniform velocity of 12 ft. per sec.: find the actual path and velocity of the ball.

42. Supposing the acceleration of gravity to vary inversely as the square of the distance from the earth's centre, find the space through which a body would fall at the distance of the moon in one hour, the radius of the earth being 4000 miles, and the distance of the moon from the earth's centre 240,000 miles.

43. A train moving at the rate of 15 miles an hour comes to the foot of an incline of 1 in 280; the friction along the plane being 8 lbs. per ton, how far will the train go before stopping?

44. Three velocities whose ratios are as $\sqrt{3}+1 : \sqrt{6} : 2$, are simultaneously impressed on a particle and the particle does not move; find the angles at which the directions of the velocities are inclined to each other.

45. A plane is 50 ft. long and of elevation 30°, the limiting angle of resistance between the plane and a given body is 15°; determine the velocity the body must have at the foot of the plane so as just to reach the top; find also the time of ascent.

46. The measure of the force of gravity being 32·2 when a second is the unit of time, what will be its measure when 10 secs. is the unit of time?

47. A body weighing 6 lbs. slides from rest down a rough inclined plane, of elevation 30°, through 20 feet in 4 seconds. What force acting along the plane would just sustain the body in equilibrium?

48. A man rows across a river $\frac{1}{4}$ of a mile broad in 5 minutes, always keeping his boat at right angles to the current. On reaching the opposite side he finds that he is $\frac{1}{2}$ a mile from the starting point. Find the velocity of the current.

49. Find the force which acting on a mass of 6 lbs. will cause it to describe 9 ft. horizontally in 2 secs. from rest. What will be the measure of this force if the unit of force be the weight of one ounce?

50. A particle is let fall from a point 91 ft. above the ground. After describing 36 ft. it meets with an obstruction which diminishes its velocity by one-half. Find the whole time of motion, and also the velocity the particle has when it strikes the ground.

51. A smooth tube ACB, consisting of two straight portions AC, CB, with a bend at its lowest point C, is fixed in a vertical plane, and a particle acted on by gravity starts from rest from a point P in the arm AC, and passing the bend C without change of velocity, rises in the arm CB to a point Q. Shew that PQ is horizontal, and that the spaces CP, CQ are proportional to the times in which they are described.

52. Two given weights are connected by a string passing over a smooth pulley. Prove that the resultant pressure between the string and pulley is less than it would have been if half the sum of the weights had been suspended at each end of the string.

53. Two particles slide down two straight lines, in the same vertical plane, at right angles to one another, starting simultaneously from their point of intersection; prove that their distance apart, at any time, will be equal to the distance either would have descended vertically in that time.

54. Prove that, if an insect crawl along the minute hand of a clock with a velocity equal to that of the extremity of the hand, it will pass from one end to the other in 9 min. 33 secs. nearly.

55. A particle has two velocities, $3u$ in a direction from A to B, and $5u$ in a direction from C to A, ABC being an equilateral triangle. Find the magnitude of its resultant velocity.

56. The tension of the connecting string when P hanging freely draws $2W$ along a smooth horizontal table is to the tension when P draws W along the table as $3:2$; compare the weights P and W.

57. AB is a quadrant of a circle whose centre is O, the radius OB being horizontal, C is a point on the quadrant, and the angle $BOC=\theta$. Shew that the time in sliding down the chord AC is to that of sliding down the chord CB as $\sqrt{\cos\frac{\theta}{2}} : \sqrt{\sin\frac{\theta}{2}}$.

58. If AC be the height and AB the length of a smooth inclined plane, prove that the time of sliding down it varies as $\frac{AB}{\sqrt{AC}}$.

59. If AC be the horizontal diameter of a vertical circle, prove that the time down any chord AB varies inversely as the time down the chord CB.

60. A particle projected vertically upwards loses half its velocity in rising through h feet. How high will it rise?

61. Two bodies are projected vertically upwards each with a velocity of 128 ft. per second, one starting 2 secs. after the other: when and where do they meet?

62. The sound of a body striking the water in a well reaches the top $3\frac{2}{4}\frac{1}{0}$ seconds after it is dropped: assuming 1120 ft. per sec. to be the velocity of sound, find the depth of the well.

63. A stone dropped from a balloon rising with a uniform velocity of 8 ft. per sec. reaches the ground in 10 secs.: find the height of the balloon when the stone was dropped.

64. A body weighing 64 lbs. is moved by a constant pressure which acting for one sec. generates a velocity of 3 ft. per sec. in the body; find the weight which the pressure would support, taking gravity at 32 ft.-sec. units.

65. A string with a heavy particle at the free end is just wound round a lamina in the shape of a regular hexagon (side a) lying on a smooth horizontal plane. If the particle be projected with velocity v at right angles to the string, find how long it will take to unwind the string.

66. If O be a fixed point, C a point in a vertical plane through O; find the locus of C when the time of sliding down CO varies as the length of CO.

67. OX is a horizontal line, C a fixed point in it, OY is drawn in any direction in a vertical plane through OC. Find by geometrical construction the point in OY from which the time of sliding down to C in a straight line is least.

68. Two straight lines of railway meet at right angles. A train starts from the junction on one line, and at the same instant another train starts towards the junction from a station on the other line, and they move at the same given speed. Find their distance apart at any time, and prove that they are nearest to each other when they are equally distant from the junction.

69. A railway train is moving at the rate of 28 miles an hour when a pistol shot strikes it horizontally in a direction making an angle $\sin^{-1}\frac{3}{5}$ with the train. The shot enters a compartment of one of the carriages at the corner farthest from the engine and passes out at the diagonally opposite corner, the compartment being 8 feet long, i.e. from window to window, and 6 ft. wide. Prove that the shot is moving at the rate of 80 miles an hour, and that it traverses the carriage in $\frac{5}{44}$ of a second.

70. Bodies slide down a number of smooth inclined planes of the same height. Prove that the velocities acquired in the descent are the same for all the bodies, but that the times of descent are proportional to the cosecants of the inclinations of the planes to the horizon.

71. Two unequal weights are connected by a string passing over a rough pulley. If the effect of friction be to prevent motion until the tension of the string at one end be greater than the tension at the other by $\frac{1}{n}$th part of the latter tension, prove that the effect on the acceleration will be the same as if the pulley remained smooth and the smaller weight were increased by $\frac{1}{n}$th of itself.

72. A man fires at an object moving with velocity u in a given direction. The velocity of the ball is such that it would have hit the object, if at rest, in one second. How far ahead of its apparent position must he aim to hit the object?

73. A weight Q rests on a smooth table, and is connected by a string passing over a pulley at the edge of the table, with a weight P hanging vertically. If after the weights have moved through a space s, Q arrives at a rough part of the table (coeff. of friction $= \mu$), find how much further they will move before stopping (if they do so).

74. A railway train passes from one station to another, a miles distant, starting with the uniform acceleration f, and when steam is shut off, and the break applied, slowing up with the uniform retardation f'. Find the time taken, f and f' being expressed in ft.-sec. units.

75. Bodies are let fall down a number of smooth inclined planes having a common vertex: prove that the locus of the points at which they all have the same given velocity is a horizontal plane.

76. A weight of 200 lbs. is to be raised through a height of 40 feet by a cord passing over a fixed smooth pulley; it is found that a constant force (P) pulling the cord at its other end for three-fourths of the ascent communicates sufficient velocity to the weight to enable it to reach the required height; find P.

77. A particle starts from A and moves for 4 secs. under an acceleration which gives it in that time a velocity of 30 ft. per sec.; it then travels uniformly with the velocity it has acquired for 3 seconds; its speed is then uniformly retarded so that in 10 secs. it comes to rest at B. Find the distance from A to B.

78. A body sliding from rest down a rough inclined plane of elevation 30°, moves over 36 ft. in the 5th second of its motion: find the coefficient of friction.

79. A circle of radius a, and a straight line inclined to the vertical at an angle $\frac{\pi}{2}-\alpha$ and lying completely outside the circle, are in the same vertical plane; prove that the time down the line of quickest descent between the circle and the straight line is $\left\{\frac{2(p-a)}{g}\right\}^{\frac{1}{2}} \sec\frac{\alpha}{2}$, p being the distance of the centre of the circle from the straight line.

80. A body projected vertically downwards overtakes in t secs. another body which has already fallen h ft. from rest from the same point; find the initial velocity of the projected body.

81. Compare the accelerations 15 mile-hour units, and 3 ft.-sec. units.

82. A particle moves from rest down a rough plane inclined at an angle 2θ to the horizon, $\tan\theta$ being the coefficient of friction. Prove that in moving over a length s of the plane it acquires the same velocity as in falling freely through a distance $s\tan\theta$.

83. An engine draws a train whose weight (exclusive of the engine) is 100 tons. The power of the engine is such that when running on the level it exerts a pull of 2 tons weight on the front carriage, and the resistance due to friction etc. is 11·2 lbs. per ton. Shew that if the engine draws the same train from rest up an incline of 1 in 300 it will in one minute acquire a velocity slightly exceeding $15\frac{3}{11}$ miles per hour.

84. A mass of weight W rests on a smooth horizontal table, also of weight W, and is connected by a light string, passing over a smooth pulley at the edge of the table, with a weight $2W$ hanging freely, which is allowed to fall, the string being initially taut. If the table

does not move, find the tension of the string, and shew that the coefficient of friction between the table and the floor is not less than $\frac{1}{4}$.

85. A point describes the circumference of a circle whose radius is a with uniform velocity in time t: prove that, if a perpendicular be let fall from the moving point on any fixed diameter of the circle, and v be the velocity of the foot of this perpendicular when its distance from the centre is x,

$$\left(\frac{vt}{2\pi}\right)^2 + x^2 = a^2.$$

86. A fixed pulley lies on a smooth inclined plane, elevation 30°, the ends of the string passing over the pulley are attached to masses of 4 lbs. and 6 lbs., of which the 4 lbs. mass lies on the plane, and the 6 lbs. mass is suspended by the string over the lower edge of the plane under the action of gravity: the motion of each weight being rectilinear, find how far the 4 lbs. mass will move from rest in 2 seconds.

87. A string passing over a fixed pulley supports two pulleys of masses M_1 and M_2 respectively. Another string passing over M_1, under a fixed pulley A, and over M_2, so that the several portions between the pulleys are vertical, has attached to its ends particles of masses m_1 and m_2 respectively. The system moves from rest in a position in which the centres of M_1, M_2 are each a feet vertically above the horizontal plane through the centre of A, and the particle m_1 is just at the end of the horizontal diameter of M_1. Find the tension of each string during the motion, and shew that, when m_1 reaches the horizontal plane through the centre of A, each of the moveable pulleys will have passed through a vertical space

$$\frac{M_1 \sim M_2}{M_1 + M_2} \cdot \frac{m_1 + m_2}{m_1 - m_2} a.$$

88. AP, AQ are two inclined planes of which AP is rough, the coefficient of friction being equal to tan PAQ, and AQ is smooth, AP lying above AQ: shew that if bodies descend from rest at P and Q they will arrive at A (1) in the same time if PQ be perpendicular to AQ, (2) with the same velocity if PQ be perpendicular to AP.

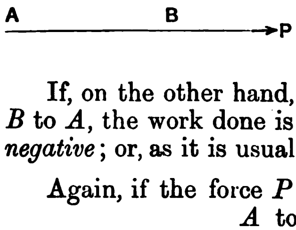
CHAPTER X.

WORK AND ENERGY.

70. **Work. A force is said to do work when it moves its point of application in the direction in which the force acts.**

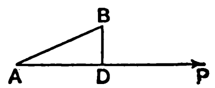
The work done is measured by the product of the force and the distance moved through in the direction in which the force acts.

Thus if a force P poundals, acting in a direction AB, move its point of application through AB feet, the work done is measured by $P.AB$ foot-poundals.



If, on the other hand, the point of application move from B to A, the work done is still $P.AB$ foot-poundals, but it is *negative*; or, as it is usually expressed, work is done *against* P.

Again, if the force P move its point of application from A to B, where B is not in the line of action of P, draw BD perpendicular to the line of action of P. The work done by P is then measured by $P.AD$, i.e. by the product of P and *the distance in the direction of the force* P through which the point of application moves.



71. **The British absolute unit of work is the work done by one poundal in moving its point of application through one foot; and is called a Foot-poundal.**

The unit of work used by engineers is the work done in lifting one pound-weight vertically through one foot, and is called a foot-pound.

Strictly speaking, this should be called a *foot-pound-weight.*

Since a pound-weight $= g$ poundals, the engineers' unit $= g$ absolute units.

The Scientific, or C.G.S., unit of work is the work done by one dyne in moving its point of application through one centimetre, and is called an Erg.

Illustrations.

A locomotive does work in pulling railway carriages along the lines.

A horse does work in pulling a barge along a canal.

A man does work in lifting a weight.

We must notice that no work is done unless the force moves its point of application *in the direction of the force:* e.g. the resistance of a smooth table does no work as a body slides along the table, for the body does not move *in the direction of the force.*

When a ring slides on a smooth rod, no work is done by the resistance of the rod, for it is always at right angles to the direction of motion, i.e. the ring, at the instant, has no motion *in the direction of this force.*

72. Rate of doing work.

The practical value of an engine or machine depends not only on the *amount* of work it can do, but also on the *time* occupied in doing it.

Thus a horse could pull a half-ton load over a distance of 7 miles, but an ordinary locomotive would do the same in a much shorter time.

We therefore need the following definition.

Power. The power of an agent is the rate at which it can do work.

Engineers have adopted as the unit of power, **one Horse-power.**

One Horse-power (H.-P.) *is the power of doing* 33000 *foot-pounds a minute or* 550 *foot-pounds a second.*

[James Watt considered that a horse could do this amount of work in a minute, but it is rather more than an ordinary horse can do.]

Thus we see that an engine of two horse-power will lift 66,000 lbs. wt. vertically through one foot in one minute or 1100 lbs. wt. vertically through one foot in one second.

73. Ex. i. *An engine starts a train of mass* 150 *tons from rest with an acceleration of* 12 *ft.-sec. units. Find the average horse-power exerted by the engine during the first second of the motion, neglecting frictional and atmospheric resistances.*

The moving force

$$=150 \times 2240 \times 12 \text{ poundals.} \qquad (P=mf)$$

Space described by the train in one second

$$=\tfrac{1}{2} \times 12 = 6 \text{ feet.} \qquad (s=ut+\tfrac{1}{2}ft^2)$$

Therefore work done in one second

$$=150 \times 2240 \times 12 \times 6 \text{ ft.-poundals}$$

$$=\frac{150 \times 2240 \times 12 \times 6}{32} \text{ ft.-lbs.}$$

Therefore the required H.-P.

$$=\frac{150 \times 2240 \times 12 \times 6}{32 \times 550}$$

$$=\frac{15120}{11}=1374\tfrac{6}{11}.$$

Ex. ii. *Find the* H.-P. *of an engine which will draw a train of mass* 200 *tons up an incline of* 1 *in* 120 *at a uniform speed of* 15 *miles per hour, the resistance due to friction etc. being* 10 *lbs. wt. per ton.*

[In order to thoroughly understand the following, the student should construct a figure.]

The retarding force due to the weight of the train

$$=\text{component of the weight down the plane}$$

$$=200 \times 2240 \cdot g \times \frac{1}{120} \text{ poundals.}$$

The retarding force due to friction etc.

$$=10 \times 200g \text{ poundals.}$$

Therefore the total retarding force

$$=\frac{200 \times 2240 \times g}{120}+2000g$$

$$=200g\left[\frac{56}{3}+10\right]$$

$$=\frac{200 \times 86 \times g}{3} \text{ poundals.}$$

A velocity of 15 miles per hour = 22 ft. per second.

Therefore the work done in each second

$$=\frac{200 \times 86 \times g}{3} \times 22 \text{ ft.-poundals}$$

$$=\frac{200 \times 86 \times 22}{3} \text{ ft.-lbs.}$$

Therefore the required H.-P.

$$=\frac{200 \times 86 \times 22}{3 \times 550}$$

$$=\frac{688}{3}=229\tfrac{1}{3}.$$

Ex. iii. *Find the speed at which an engine of 500* H.-P. *can draw a train of 120 tons mass up a slope of 1 in 100, the frictional resistance being $5\frac{1}{10}$ lbs. wt. per ton.*

[The student should draw a figure.]

Let v feet per sec. be the velocity required.

The retarding force due to the weight of the train

= component of the weight along the plane

$$=\frac{120 \times 2240g}{100} \text{ poundals.}$$

The retarding force due to friction

$$=\frac{51}{10} \times 120g \text{ poundals.}$$

Therefore total retarding force

$$=120g\left[\frac{51}{10}+\frac{2240}{100}\right] \text{ poundals}$$

$$=12\times 275g \text{ poundals.}$$

Therefore $v\times 12\times 275g$ ft.-poundals

$$=\text{work done in one second}$$

$$=550\times 500g.$$

Hence $$v=\frac{250}{3}\text{ ft. per second}$$

$$=\frac{250\times 60\times 60}{3\times 1760\times 3}\text{ miles per hour}$$

$$=\frac{625}{11}=56\tfrac{9}{11}\text{ miles per hour.}$$

74. The **efficiency** or **modulus** of an engine, or machine, is the ratio of the actual work done on the *weight* to the work done by the *power*; or

$$\text{efficiency}=\frac{\text{work done on weight}}{\text{work done by power}}.$$

In every machine some work done by the power is wasted in overcoming frictional and other resistances, so that the efficiency of a machine is always less than unity.

If an engine of efficiency μ works at 30 H.P., the *useful work* it can do is $30\times 550\mu$ ft.-lbs. per second.

75. Ex. i. *A steam-crane lifts a weight of* 8 *tons steadily through a height of* 11 *feet in* 48 *seconds. If the efficiency of the engine is* ·4, *at what* H.P. *is it working?*

Let x be the required H.P.

Then $$x\times 550\times 48\times \cdot 4=\text{the work done on the weight}$$

$$=8\times 2240\times 11.$$

Whence $$x=18\tfrac{2}{3}\text{ H.P.}$$

Ex. ii. *A heavy uniform chain of mass* 300 *lbs. and length* 40 *feet is suspended from one end. What is the work done in lifting the chain to the point of suspension?*

We may suppose the mass of the chain collected at its centre of gravity, i.e. 20 feet below the point of suspension.

$$\therefore \text{ the work required}=300\times 20$$

$$=6000\text{ ft.-lbs.}$$

EXAMPLES. X *a*.

[It may be assumed that 1 cubic foot of water $= 62\cdot5$ lbs. wt. $= 6\cdot25$ gallons.]

1. Find the work done by a mass of 11 cwt. in falling vertically through 10 feet.

2. How long would an engine of 2 H.P. take to do the work in the previous question?

3. An engine, whose efficiency is $\cdot 3$, lifts a weight of 15 cwt. through 10 feet in 20 secs.: find its H.P.

4. An electric tramcar is drawn along a horizontal road at the rate of $7\frac{1}{2}$ miles per hour. If the resistances to motion are equal to a force of 15 lbs. wt. per ton, find the number of foot pounds of work done in a minute. What H.P. is this equivalent to?

5. A man weighing 14 stone ascends a tower 110 feet high. What work does he do in ascending? How long would an engine of $\frac{1}{2}$ H.P. take to lift him there?

6. How long would an engine of $\frac{1}{2}$ H.P., and efficiency $\cdot 4$, take to pump 1000 gallons of water from a depth of 33 feet?

7. Find the work done by a man in emptying a cistern 3 feet deep and containing 1650 gallons of water. How long would an engine of $\frac{1}{2}$ H.P., and efficiency $\cdot 6$, take to do it?

8. What horse-power is required to pull a weight of 2 tons up a smooth inclined plane of elevation 30° at the rate of 10 ft. per sec.?

9. In the preceding question, what will be the necessary horse-power if the frictional resistances to motion amount to 10 lbs. wt. per ton?

10. A load of 300 lbs. is drawn up an incline of 1 in 100, the total frictional resistance being equal to the weight of 7 lbs. If starting from rest and moving with a uniform acceleration, it acquires a velocity of 10 ft. per sec. in one minute, find the average horse-power employed during that time.

11. A uniform cylinder 8 ft. high, standing on its base of diameter 6 feet, weighs one ton. How many foot-pounds of work must be done to upset it?

12. A train running at 45 miles an hour weighs 125 tons. If the average pull on it is 11 lbs. wt. per ton, what horse-power is employed upon it? If the efficiency of the engine is $\cdot 4$, what horse-power will the engine be working up to?

13. A steam crane raises a weight of 6 tons uniformly through a height of 33 ft. in 36 seconds; find at what H.-P. it is working.

14. How much work is done in lifting a mass of 11 tons vertically through 6 feet? What is the H.-P. of an engine which will do this in 2 minutes?

15. Find the work done in pulling a mass of 5 tons 12 feet up a smooth inclined plane rising 1 in 10. Express the answer in foot-tons.

16. What volume of water will an engine of 50 H.-P. raise in 4 hours from a depth of 120 ft., if the modulus of the engine be ·5?

17. A cylindrical well of 4 ft. diameter and 100 ft. deep is half full of water. Find the H.-P. of the engine which would empty it in 20 minutes, delivering the water at the surface. ($\pi = \frac{22}{7}$.)

18. A uniform circular cylinder, 6 ft. in diameter and 8 ft. long, is lying on the ground. Find the work done in putting it up on end, if its weight be half a ton.

19. A uniform chain, 150 yards in length and weighing 3 lbs. a foot, is attached at one end to a windlass. Find the work done in winding up the chain slowly.

20. An engine drives a shaft by means of a belt running at the rate of 40 ft. per sec., and the tension of the tight portion of the belt is 120 lbs. wt. whilst that of the slack portion is 50 lbs. wt. Find the H.-P. transmitted by the belt.

21. A cylindrical hollow tower, of internal diameter 6 feet, height 100 ft., thickness of wall 3 ft., is built of brickwork weighing 50 lbs. a cubic foot. Find the number of hours in which an engine of 6 H.-P. would raise the bricks from the ground into position. ($\pi = \frac{22}{7}$.)

22. What must be the horse-power of a locomotive engine which moves at the rate of 20 miles an hour upon a level rail, the weight of the train being 50 tons, and the resistance of friction 8 lbs. wt. per ton?

23. What distance will an engine of $82\frac{8}{11}$ horse-power drag a train in ten minutes, with a uniform velocity, up an incline of 1 in 200, the weight of the train being 80 tons, and the friction 7 lbs. per ton?

24. What must be the horse-power of a locomotive engine to draw a gross load of 40 tons wt. at the rate of 30 miles an hour along a horizontal road, the resistances being estimated at 8 lbs. wt. per ton?

25. An engine of 80 H.-P. makes 60 revolutions a minute, and the stroke of its piston is 3 feet: find the mean pressure on the piston, the modulus of the engine being ·4.

26. A right-angled triangle ABC turns stiffly in its own plane about the middle point of the hypotenuse AB. Forces just sufficient to overcome the resistance are applied thus:—P at B at right angles to BC, Q at C in direction BC, R at A at right angles to AB, all the forces tending to turn the triangle in the same direction: shew that the work done in turning the triangle through a right angle $= \frac{1}{2}\{Rc + (P - Q)(a - b)\}$; the forces remaining throughout the motion parallel to their original directions and constant in magnitude.

27. At the bottom of a coal mine 275 ft. deep there is an iron cage containing coal, weighing 14 cwt., the cage itself weighing 4 cwt. 109 lbs., and the wire rope that raises it 6 lbs. per yard. Find the work done when the load has been lifted to the surface and the horse-power required to do that amount of work in 40 seconds.

28. A locomotive draws a load of m lbs. up an incline the angle of which is a, the coefficient of friction being μ. If, starting from rest and moving with uniform acceleration, it acquires a velocity v in t seconds, find the average horse-power at which it has worked during that time.

29. The diameter of the piston of a steam-engine is 80 inches, the mean pressure of the steam 24 lbs. wt. per square inch, the length of the stroke 10 ft., the number of strokes per minute 11. How many cubic feet of water will it raise per minute from a depth of 33 fathoms, the modulus of the engine being ·7? $(\pi = \frac{22}{7}.)$

30. Three equal masses of 10 lbs., connected by strings knotted at A, hang in equilibrium over two smooth pegs B and C, $10\sqrt{3}$ feet apart in a horizontal line. Find the work done in moving the point A to the point B.

31. The diameter of a cylinder of a steam-engine is 80 inches, the piston makes 8 strokes of $12\frac{1}{4}$ feet a minute, under a mean pressure of 15 lbs. the square inch, the modulus of the engine is ·56; how many cubic feet of water will it raise in one minute from a depth of 224 feet?

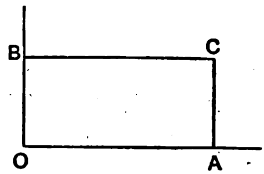
32. In a steam-engine pumping water from a mine, what must be the length of the stroke of the piston, the surface of which is 1500 sq. inches, and which makes 20 strokes a minute, so that with a mean pressure of 12 lbs. wt. on each sq. in. of the piston, the engine may be of 80 horse-power?

33. A cistern 54 ft. long, 44 feet wide, and 8 ft. deep is to be filled with water from a well, the surface of which always stands at 36 ft. below the bottom of the cistern: in what time will an engine of 2 horse-power do it?

76. Geometrical representation of Work done.

(1) *When a constant force P moves a body through a space s in its own direction.*

Let OA represent the space s, and OB, at right angles to OA, the force P. Complete the rectangle $AOBC$. Then the area of the rectangle represents Ps the work done.



(2) *When the force P is variable.*

Take Os as space-axis, and OP, at right angles to it, as force-axis.

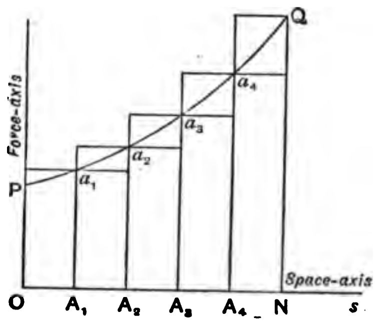
Let ON represent space s, and divide it into n equal parts at A_1, A_2, ...

At A_1, A_2, ... draw ordinates A_1a_1, A_2a_2, ... to represent the corresponding values of P to the spaces OA_1, OA_2, ...

Complete the rectangle as shown in the diagram, and join a_1, a_2, ... by an even curve.

Then, as in Art. 19, it may be shown that the work done in moving the body through space s, lies *between* the sum of the greater rectangles and the sum of the less.

Hence making n indefinitely large, the work done will be represented by the area $PONQ$.



77. *To represent geometrically the work done in moving a body slowly through a space s by means of a force which is proportional to the space passed over.*

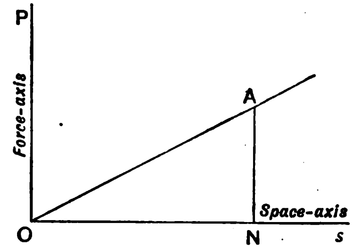
Taking Os for space-axis, and OP, at right angles to it, as force-axis, let ON, and NA represent corresponding values of s and the force P. AN is proportional to ON.

$\therefore$ the locus of A is a straight line through O.

$\therefore$ the work done by the force in moving the body slowly through space s is represented by the area of the triangle OAN

$$= \tfrac{1}{2}AN \, . \, ON = \tfrac{1}{2}P \, . \, s.$$

The above method will be found useful in finding the work done in slowing stretching an elastic string, or in slowly compressing, or stretching, a spiral spring.



Elastic Strings. Hooke's Law.

The tension of an elastic string varies as its extension beyond its unstretched length.

EXAMPLES. X *b*.

1. An elastic string is stretched through one inch by a force of 6 lbs. wt. Find the work done in slowly stretching it through 3 inches.

2. A force of 40 lbs. wt. is required to slowly compress a spiral spring through one inch. Find the work done in compressing it through 5 inches.

3. A uniform chain AB 20 ft. long, and weighing 50 lbs., hangs vertically, suspended from A. Draw a diagram to represent the work done in slowly lifting the end B to the end A. Find the amount of work done.

4. A chain 200 ft. long and weighing 3 lbs. a foot hangs suspended from one end to a windlass. Draw a diagram to show the amount of work done in rolling the chain up on the windlass, and find the amount of work in ft.-lbs.

78. Energy. The energy of a body is its capacity for doing work.

Any moving body has the power of overcoming resistance whilst its velocity is being destroyed; therefore every moving body has the capacity of doing work, i.e. it possesses energy.

Again, some bodies *when at rest* possess a capacity for doing work (i.e. they possess energy) *by virtue of their position.* Thus a stone held at a height from the ground will do work if it is allowed to fall: a coiled-up clock spring is capable of doing work.

Hence we have two kinds of energy.

The Kinetic energy of a body is the energy it possesses by virtue of its motion. It is sometimes called **vis-viva.**

The Potential energy of a body is the energy it possesses by virtue of its position.

We may look upon the Potential Energy of a body as the energy stored up in it; and we shall see that there is no Potential Energy unless work has previously been done on the body.

79. Measurement of Kinetic Energy.

The kinetic energy of a body is measured by the work which it is capable of performing against impressed forces whilst its velocity is being destroyed.

Let a body, mass m, moving with velocity u, be brought to rest by a uniform force P. Let f be the retardation produced by this force, and let the body move through space s in coming to rest.

Then $$P = mf,$$

and $$0 = u^2 - 2fs, \qquad (v^2 = u^2 + 2fs)$$

i.e. $$s = \frac{u^2}{2f}.$$

Therefore the kinetic energy of the body

$= $ the work done whilst the body is being reduced to rest

$= P \,.\, s$

$= mf \,.\, \dfrac{u^2}{2f} = \frac{1}{2}mu^2.$

The kinetic energy of a body is therefore equal to half the product of its mass and the square of its velocity.

80. Measurement of Potential Energy.

The Potential Energy of a body is measured by the work it is capable of doing in moving from its initial position to some standard position.

Thus in the case of a body of mass m possessing potential energy in virtue of its position at a height h from the ground, the energy is measured by the work done by the body in coming to the ground and is therefore equal to mgh.

Now the work done in lifting the body to height $h = mgh$; hence we see that the

Potential energy of a body (the energy stored up in it) $=$ *the work which has been done on the body.*

81. *The change of kinetic energy of a body is equal to the work done on it.*

Let a force P act on a mass m, and change its velocity from u to v whilst the body moves through space s. Let f be the acceleration produced, so that $P = mf$.

Then the change in the kinetic energy of the body

$$= \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2$$
$$= \tfrac{1}{2}m(v^2 - u^2)$$
$$= \tfrac{1}{2}m \,.\, 2fs \qquad (v^2 = u^2 + 2fs)$$
$$= mfs$$
$$= Ps$$
$$= \text{the work done on the body.}$$

82. *For a falling body the sum of the kinetic and potential energies is constant.*

Let m be the mass of the body, h its original height from the ground, x its height at the instant during its fall when its velocity is v.

When the body is at height x, the sum of its kinetic and potential energies

$$= \tfrac{1}{2}mv^2 + mgx$$
$$= \tfrac{1}{2}m \,.\, 2g(h - x) + mgx, \text{ for } v^2 = 2g(h - x)$$
$$= mgh$$
$$= \text{a constant quantity.}$$

This is a very simple case of what is known as the *Principle of the Conservation of Energy.*

83. Principle of Conservation of Energy.

On this point one cannot do better than read what Professor Clerk Maxwell says in his small book on "**Matter and Motion.**" He says: "*When the nature of a material "system is such that if, after the system has undergone any "series of changes, it is brought back in any manner to its "original state, the whole work done by external agents on the "system is equal to the whole work done by the system in over-"coming external forces, the system is called a* **Conservative "System.**

"The progress of physical science has led to the discovery "and investigation of different forms of energy, and to the

"establishment of the doctrine that all material systems may "be regarded as conservative systems, provided that all the "different forms of energy which exist in these systems are "taken into account.

"This doctrine, considered as a deduction from observa- "tion and experiment, can, of course, assert no more than "that no instance of a non-conservative system has hitherto "been discovered.

"As a scientific or science-producing doctrine, however, it "is always acquiring additional credibility from the constantly "increasing number of deductions which have been drawn "from it, and which are found in all cases to be verified by "experiment.

"**General statement of the Principle of Conserva- "tion of Energy.**

"**The total energy of any material system is a "quantity which can neither be increased nor dimin- "ished by any action between the parts of the system, "though it may be transformed into any of the forms "of which energy is susceptible.**

"If, by the action of some agent external to the system, "the configuration of the system is changed, while the forces "of the system resist this change of configuration, the external "force is said to do work on the system. In this case the "energy of the system is increased by the amount of work "done on it by the external agent.

"If, on the contrary, the forces of the system produce a "change of configuration which is resisted by the external "agent, the system is said to do work on the external agent, "and the energy of the system is diminished by the amount "of work which it does.

"Work, therefore, is a transference of energy from one "system to another; the system which gives out energy is "said to do work on the system which receives it, and the "amount of energy given out by the first system is always "exactly equal to that received by the second.

"If, therefore, we include both systems in one larger "system, the energy of the total system is neither increased

"nor diminished by the action of the one partial system on "the other."

84. Ex. i. *A train is moving on a horizontal rail at 30 miles an hour: if the steam be suddenly turned off, how far will it run before it stops, the resistances being taken at 8 lbs. wt. per ton?*

$$30 \text{ miles an hour} = \frac{30 \times 1760 \times 3}{60 \times 60} = 44 \text{ ft. per sec.}$$

Let m tons be the mass of the train, and x feet the distance required.

The total resistance $= 8mg$ poundals.

The work done on the train after the steam is shut off

$$= \text{kinetic energy destroyed.}$$

$$\therefore\ 8mg \, . \, x = \tfrac{1}{2} \, . \, m \, . \, 2240 \, . \, (44)^2, \qquad (Ps = \tfrac{1}{2}mv^2)$$

$$\therefore\ x = \frac{1120 \, . \, (44)^2}{8 \, . \, 32} \text{ feet}$$

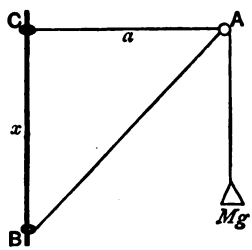
$$= 70 \times 121 \text{ feet}$$

$$= \frac{70 \times 121}{5280} \text{ miles}$$

$$= \frac{77}{48} = 1\tfrac{29}{48} \text{ miles.}$$

Ex. ii. *A heavy ring, of mass m, slides on a smooth vertical rod, and is attached to a light string, which passes over a smooth peg distant a from the rod, and which has a mass M (>m) fastened to its other end. The ring is dropped from a point in the rod in the same horizontal plane as the peg: find what distance it will descend before coming to rest.*

Let A be the peg, C the initial position of the ring, B its position at rest when it has descended through space x on the vertical rod CB.



The work done on the ring

$$= mgx.$$

The work done on the mass M

$$= Mg\,(AB - a)$$

$$= Mg\,(\sqrt{a^2 + x^2} - a).$$

But the work done on the ring = the work done on the mass M.

$$\therefore\ mgx = Mg\left(\sqrt{a^2+x^2}-a\right),$$

$$mx + Ma = M\sqrt{a^2+x^2},$$

$$m^2x^2 + 2mMax + M^2a^2 = M^2a^2 + M^2x^2.$$

$$\therefore\ x = \frac{2Mma}{M^2-m^2}.$$

ENERGY. EXAMPLES X *a*.

1. How many British absolute units of energy are there in an oz. bullet moving with a velocity of 1200 ft. per second?

To what height must it be carried to have the same amount of potential energy?

2. A bullet moving with a velocity of 1000 ft. per sec. loses 20 per cent. of its velocity in passing through a plank: how many such planks will it pass through if they are placed together?

3. How many foot-pounds of work are required to change the velocity of a body of P lbs. weight from v to v' ft. per second?

4. A five-ton hammer falls through 8 ft., and thereby compresses a mass of iron $\frac{1}{4}$ in., the resistance of the iron being uniform throughout the compression. What steady pressure exerted by a hydraulic press would produce the same effect?

5. A truck runs from rest down an incline of 1 in 100 for a distance of 1 mile: it then runs up an equal gradient with its acquired velocity for a distance of 500 yds. before stopping. Assuming that it loses no velocity in moving from one incline to the other, find the total resistance, frictional or other, in pounds weight per ton, which has been opposing the motion.

6. Find the kinetic energy after 2 secs. of a body of mass 20 lbs. projected vertically upwards with a velocity of 128 ft. per second.

7. A body of mass 10 lbs. is projected vertically upwards with a velocity of 256 ft. per sec.; find its kinetic energy when it is at a height of 64 feet.

8. A body weighing 10 lbs. slides down an inclined plane whose height is 25 feet; it reaches the foot of the plane with a velocity of 30 ft. per sec.: how many foot-pounds of energy have been expended during the motion on friction and other resistances?

9. An engine pumps water through a hose, and the water leaves the hose with a velocity v: shew that kinetic energy is imparted to the water at a rate which varies as v^3.

10. A bicyclist, working at the rate of one-twelfth of a horse-power, goes 10 miles an hour on a level road: find the resistance of the road.

11. The mass of a fly-wheel is 10,000 lbs. and it is of such a size that it may all be supposed concentrated on the circumference of a circle 12 ft. in radius: what is its kinetic energy, in ft.-lbs., when making 15 revolutions a minute? ($\pi = \frac{22}{7}$.)

12. How many turns would the fly-wheel in the previous question make before coming to rest if the steam were cut off and it moved against a friction of 400 lbs. wt. exerted on the circumference of an axle 1 ft. in diameter? ($\pi = \frac{22}{7}$.)

13. A railway train is moving on a horizontal road at the rate of 30 miles an hour, when the steam is suddenly turned off; how far will the train run before it stops, the resistances being estimated at 7 lbs. wt. per ton?

14. A truck runs from rest down an incline of 1 in 100 for a distance of 1740 yards; and then runs up an equal gradient with its acquired velocity for a distance of 500 yds. before stopping. Assuming the principle of work, find the total resistance, frictional or other, in lbs.-wt. per ton, which has been opposing the motion.

15. A bullet with an initial velocity of 1500 ft. per sec. strikes a target 1200 yds. distant with a velocity of 900 ft. per sec., the range of the bullet being assumed to be horizontal: compare the mean resistance of the air with the weight of the bullet.

16. An engine whose efficiency is ·4 projects 1000 gallons of water a minute with a velocity of 88 ft. per second. Find its H.-P.

17. If a particle whose mass is 4 ounces has a velocity of 1000 ft. per sec., through what distance will it overcome a force of 360 lbs. wt. before coming to rest?

18. A particle, whose mass is 40 lbs., moves in a straight line; its velocity is diminished from 50 ft. per sec. to 30 ft. per sec.; find how many foot-poundals of kinetic energy it has lost. If the loss is due to the action of a constant force, while the particle passes over 40 ft., find the value of the force in lbs. wt.

19. A body, whose weight is capable of exerting a downward force of 320 poundals, is held at a height above the ground, such that its potential energy is 4,800 British absolute units of energy. If the body is allowed to fall, determine its potential and kinetic energies after it has fallen one foot.

20. Two particles, whose masses are 12 lbs. and 9 lbs., are connected by a fine thread which passes over a smooth pulley (as in Atwood's machine); if the system moves through 5 ft. from rest, find its kinetic energy in foot-poundals.

What will be the numerical value of the change in the potential energy of the system, and will it be an increase or a decrease?

21. A train of mass 150 tons, running down a smooth incline of 1 in 120, has just attained a velocity of 30 miles an hour: find the resistance necessary to stop the train in a quarter of a mile.

22. What is the H.-P. of an engine which can project 10,000 lbs. of water per minute with a velocity of 80 ft. per sec., twenty per cent. of the whole work done being wasted by friction etc.?

23. A blacksmith wielding a 14 lb. sledge strikes an iron bar 25 times a minute, bringing the sledge to rest upon the iron at each blow. If the velocity of the sledge on striking the iron be 32 ft. per sec., compare the rate at which the smith works with a horse-power.

24. Find the work done per second by a waterfall 30 yds. high, a quarter of a mile broad, where the mass of water is 20 ft. deep, and has a velocity of $7\frac{1}{2}$ miles per hour when it arrives at the fall.

25. A heavy wheel of 5 ft. radius revolves on a horizontal axle 2 inches in diameter, coefficient of friction $\frac{1}{5}$, making 10 turns a minute; its mass being supposed collected in the rim. If left to itself, how many revolutions will it make before stopping?

26. A chain 300 ft. long and weighing 3 lbs. a foot is wound on a windlass. Find the difference of its potential energy in this position, and in its position when 200 ft. of the rope are unwound, neglecting friction and the weight of the roller, and supposing that no part of the rope touches the ground.

27. A train whose mass is 60 tons starts from rest at the top of an incline of 1 in 100. After it has gone 400 yards its velocity is 12 yds. per sec. Find (neglecting the effect of friction) the work done by the engine in foot-pounds.

28. A mass of half-a-pound falls from rest upon some dough, the surface of which is 18 inches below the starting point, and it penetrates 3 inches. What is the work done on the dough? and what is the average pressure which it has exerted?

29. Find the number of foot-pounds of work done by a man, who picks up a stone of mass 4 lbs. and throws it with a velocity of 48 ft. per sec.; the man's hand, at the instant when the stone leaves it, being 6 ft. above the ground.

30. A bicyclist who works at a uniform rate and whose mass together with that of his machine is 175 lbs., rides at the rate of 30 ft. per sec. on the level and 20 ft. per sec. up an incline of 1 in 100. That part of the force opposing the motion of the machine which is independent of the inclination of the road being assumed to be constant and equal to F, determine the constant velocity down an incline of 1 in 200, and prove that F is equal to $3\frac{1}{2}$ lbs. wt.

31. The roughness of an inclined plane is just sufficient to prevent a body placed on it from sliding down. Prove that the work done in

pushing the body up the plane would be double of that done in lifting it through the vertical space by which it is raised.

32. Find the tensions of the coupling of a locomotive (1) when it is drawing a train 30 miles an hour and doing work upon it at the rate of 100 horse-power; (2) when it is generating an acceleration of $\frac{1}{2}$ a foot per sec. per sec. in a train of 80 tons reckoning the resistance due to friction etc. at 10 lbs. wt. per ton.

33. Find the amount of work accumulated in ft.-tons in a train of 120 tons mass when its velocity is sufficient to carry it a quarter of a mile up an incline of 1 in 100, the resistances on level road being equal to a weight of 8 lbs. per ton.

34. Prove that a train travelling at 45 miles an hour will be brought to rest in about 284 yds. by the brakes, supposing them to press on the wheels with half the weight of the train, and that the coefficient of friction is 0·16.

35. A ball of mass m after falling through h feet strikes a mud-bank and penetrates it to a depth s ft. before being brought to rest. Prove that the work done on the ball by the pressure of the mud is equal to

$$mg(h+s).$$

36. In the previous question find the work done by the pressure of the mud if the ball strikes it with the same velocity but in a horizontal direction.

37. A weight of 6 lbs. is placed upon a rough horizontal table, and is moved along the table by a weight of 4 lbs. which hangs over the edge and is attached to the 6 lb. mass by a light inelastic string. Find the kinetic energy (in ft.-lbs. to two significant digits) of the system after it has moved for 3 secs. from rest, the coefficient of friction being 0·4.

38. A steam hammer weighing 6 tons falls through 4 feet from rest. Find the steady force necessary to bring it to rest in the next 3 inches.

39. A 2 oz. bullet, moving horizontally with a velocity of 1200 ft. per sec., strikes a sand-bank and penetrates to a depth of 6 inches. Find the average resistance (in lbs. wt.) of the sand to penetration.

40. A man having a light rope attached to a mass m lbs. at the bottom of a shaft h ft. deep, pulls with a steady force until the mass is half way up, and then ceases. The mass just arrives at the top of the shaft: find the force with which the man pulls.

41. A given weight W is hoisted up to a given height h by a workman. During the first third of the height the weight is made to move upwards with a constant acceleration, during the second third

with a constant velocity, during the last third it is brought gradually to rest with a constant retardation. Supposing that the resistances (including gravity) are represented by a constant retarding force R, and that the whole time of the ascent is equal to that of falling from rest through a height nh, prove that the difference of the works done by the man in the first and third portions of the ascent is

$$\frac{25\,Wh}{18n}.$$

State also the whole work done.

CHAPTER IX.

RELATIVE MOTION.

85. WHEN the distance between two points changes in magnitude or in direction, or in both, the points are said to have **Relative velocity.**

Thus a person seated in a moving railway carriage has velocity relative to the station he is approaching or leaving but none relative to the carriage in which he is seated.

When two points move in the same straight line with different velocities, the distance between them changes in magnitude but not in direction, and the points have relative velocity.

When a point moves on the circumference of a circle, the line joining it to the centre changes in direction but not in magnitude, and the point therefore has velocity relative to the centre.

We must also remember that all motion is relative. Thus when we talk of a man walking at the rate of 4 miles an hour, we mean that he is moving at that rate relative to the earth.

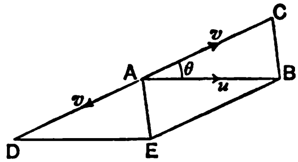
If two points are in motion it is most important to notice that their relative motion is unaltered if the same additional velocity or acceleration be impressed upon both.

Thus in the case of the motion of two points on the earth's surface, their relative motion is the same whatever the speed of rotation of the earth may be about its axis;

nor will it vary with the speed of the earth in its annual path round the sun.

86. *Two particles are moving in given directions with given velocities; to find their relative velocity.*

Let AB, AC represent the velocities u and v of the particles, and let $\angle BAC = \theta$.



Give to each particle a velocity equal but opposite to the velocity v along AC.

Since we do the same to both particles this does not alter their relative velocity.

The particle moving in the direction AC is now reduced to rest, and the other particle has velocity u along AB, and velocity v along CA produced.

Produce CA to D making AD equal to AC, and complete the parallelogram $ADEB$.

By the parallelogram of velocities, AE is the resultant velocity of the second body, and the first body being now at rest AE represents the relative velocity required;

$$AE^2 = AD^2 + AB^2 + 2AD \, . \, AB \cos DAB$$
$$= u^2 + v^2 - 2uv \cos \theta,$$
$$\therefore \; AE = \sqrt{u^2 + v^2 - 2uv \cos \theta}.$$

Join BC. Then CB is parallel and equal to AE, for EB is equal and parallel to AC.

Hence if AB, AC, drawn from the point A represent two velocities, BC represents their relative velocity.

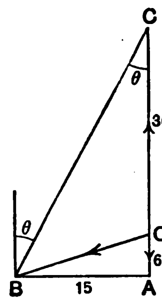
Also BC represents the velocity of the point C relative to the point B, and CB represents the velocity of the point B relative to the point C.

Cor. We see that if u and v be in the same straight line, or parallel, the relative velocity becomes $u \pm v$, according as the points are moving in opposite or the same directions.

87. By substituting the word 'acceleration' for the word 'velocity' throughout the previous article, we obtain the relative acceleration of two particles moving with given accelerations.

88. Ex. i. *A ship steams due west at the rate of* 15 *miles an hour relative to a current which is flowing at the rate of* 6 *miles an hour due south. What is the velocity relative to the ship of a train going due north at the rate of* 30 *miles an hour?*

Draw OA due south from O, 6 units in length, to represent the velocity of the current; AB due west from A, 15 units in length, to represent the velocity of the ship relative to the current. Join OB. Then OB represents the actual velocity of the ship. (Art. 86.)



Again draw OC due north from O, 30 units in length, to represent the velocity of the train; then by the same article BC represents the velocity of the train relative to the ship.

Also $$BC^2 = AB^2 + AC^2 = 15^2 + 36^2$$
$$= 3^2[5^2 + 12^2]$$
$$= 3^2 . 13^2,$$
$$\therefore BC = 39.$$

And if $\angle BCA = \theta$, $\qquad \tan\theta = \dfrac{AB}{AC} = \frac{15}{36} = \frac{5}{12}.$

Therefore the velocity of the train relative to the ship is 39 miles per hour in a direction $\tan^{-1}(\frac{5}{12})$ east of north.

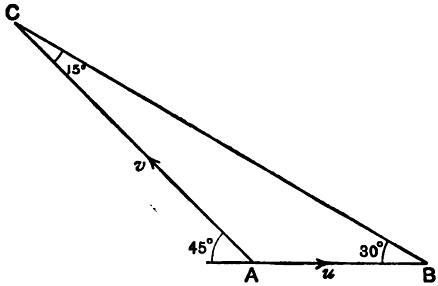
Ex. ii. *A steamer is going due east with velocity* u; *its smoke-track points* 30° *north of west. If the wind be from the south-east, find its velocity.* [See also Art. 89.]

[In problems of this nature it is assumed that the smoke, on leaving the steamer, moves off in the direction of the wind. If the motion took place in a vacuum, the smoke on leaving the steamer would retain the velocity of the steamer. The atmosphere is supposed to destroy this velocity of the smoke.]

Draw AB to represent in magnitude and direction the velocity, u, of the steamer. Let BC represent the smoke-track when the steamer is at B, so that $\angle ABC = 30°$. Draw AC making an angle of 45° with BA produced, so that AC is the direction of the wind.

Hence since BC is the smoke-track, the particle of smoke at C must have left the steamer when it was at A; i.e. this particle of

smoke has travelled from A to C whilst the steamer has travelled from A to B.



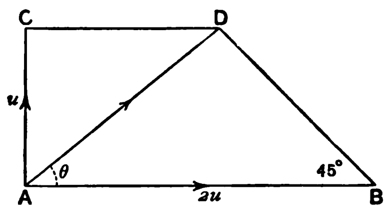
Since then AB represents the velocity of the steamer, AC represents the velocity of the smoke, i.e. of the wind. Let $AC=v$,

$$\frac{v}{u}=\frac{AC}{AB}=\frac{\sin 30^\circ}{\sin 15^\circ}=\frac{\sqrt{2}}{\sqrt{3}-1}=\frac{\sqrt{2}(\sqrt{3}+1)}{2}.$$

Therefore the velocity of the wind

$$=\frac{\sqrt{2}(\sqrt{3}+1)}{2}u.$$

Ex. iii. *On board a ship steaming due east with velocity $2u$, the wind appears to blow from the south-east; whilst on board a ship steaming due north with velocity u, it appears to blow from the west: find the actual velocity and direction of the wind.*



Let AB represent the velocity ($2u$) of the first ship, AC the velocity (u) of the second ship, AD the actual velocity of the wind. It is required to find the magnitude and direction of AD.

Join BD, CD; and let $\angle DAB=\theta$.

BD represents the velocity of the wind relative to the first ship;

$$\therefore \angle DBA=45^\circ.$$

Also CD represents the velocity of the wind relative to the second ship;

$$\therefore \angle ACD = 90^\circ.$$

From $\triangle ACD$ $\qquad AD = \dfrac{AC}{\sin ADC} = \dfrac{u}{\sin\theta}$ .... (1).

From $\triangle ADB$ $\qquad \dfrac{AD}{\sin ABD} = \dfrac{AB}{\sin ADB}$;

$$\therefore \text{from (1)} \ \frac{u}{\sin\theta \cdot \sin 45^\circ} = \frac{2u}{\sin(\theta + 45^\circ)},$$

$$\sin(\theta + 45^\circ) = 2\sin\theta \sin 45^\circ.$$

Expanding we see that $\sin\theta = \cos\theta$; $\therefore$ $\theta = 45^\circ$ and $AD = u\sqrt{2}$ from (1), i.e. the wind blows from the south-west, with a velocity $u\sqrt{2}$.

Ex. iv. *An inelastic string, passing over a smooth pulley, is attached at one end to a mass m_1, and at the other to another smooth pulley over which passes a second inelastic string carrying masses m_2, m_3 at its ends: find the tensions of the strings and the accelerations of the different masses, the masses of the pulleys and strings being negligible.*

Let $\qquad f_1$ be the acceleration of the mass m_1 upwards,

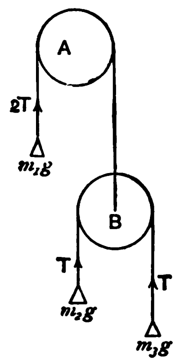
$\qquad f_2$ .... m_2 downwards,

$\qquad f_3$ .... m_3 downwards,

$\qquad T$ the tension of the string over the moveable pulley,

then $\qquad 2T$ is .... fixed pulley.

[This may be seen by considering the pulley B, whose mass by hypothesis is zero. Let T' be the tension of the upper string, then $2T - T' = 0$, since $P = mf$; i.e. $T' = 2T$.]



Considering the mass m_1, $2T - m_1 g = m_1 f_1$ $(P = mf)$.........(1),

.... m_2, $m_2 g - T = m_2 f_2$ .... (2),

.... m_3, $m_3 g - T = m_3 f_3$ .... (3).

Again, the acceleration of the mass m_2 *relative to pulley* $B = f_2 - f_1$ downwards, and the acceleration of the mass m_3 *relative to pulley* $B = f_3 - f_1$ downwards.

Also the string being inelastic these relative accelerations are equal *but opposite in direction;*

$$\therefore\ f_2 - f_1 = -f_3 + f_1,$$

i.e.
$$f_2 + f_3 = 2f_1 \quad \text{.................................(4)}$$

Hence, from (2) and (3),

$$2g - \frac{T}{m_2} - \frac{T}{m_3} = f_2 + f_3$$

$$= 2f_1 \text{ by (4)}$$

$$= \frac{4T}{m_1} - 2g \text{ by (1)},$$

whence
$$T\left[\frac{4}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right] = 4g,$$

$$\therefore\ T = \frac{4m_1 m_2 m_3 g}{4m_2 m_3 + m_1 m_2 + m_1 m_3}.$$

By substituting this value of T in equations (1), (2), (3) we determine the values of f_1, f_2, f_3; and we then have the tensions and accelerations required.

EXAMPLES. XI *a*.

1. One ship is sailing due west at 12 miles an hour, another due south at $12\sqrt{3}$ miles an hour; find, in magnitude and direction, the velocity of the first ship relative to the second.

2. A ship steams in a direction north-east with a velocity of $6\sqrt{2}$ miles per hour relative to a current running due west at 6 miles per hour. Find the actual velocity in magnitude and direction, of the ship.

3. In the preceding example find the actual velocity, in magnitude and direction, of the ship when the current runs due east.

4. Two trains, each 132 feet long, pass one another in 6 secs. when they are going in the same direction, and in 2 secs. when they move in opposite directions: find their velocities.

5. Find the apparent velocity, in magnitude and direction, of rain falling vertically with a velocity of 44 ft. per second, to a passenger in a train moving at the rate of 60 miles per hour.

6. To a man walking due west at 3 miles an hour, the wind appears to blow from the south at 4 miles an hour. Find the real velocity of the wind in magnitude and direction.

7. P and Q are two points 5 miles apart. One man starting from P walks towards Q at 3 miles an hour, whilst another starting from Q at the same instant walks at right angles to QP at 4 miles an hour: what is their shortest distance apart, and their time of reaching it?

8. To a passenger in a train, raindrops seem to be falling at an angle of 45° to the vertical: they are really falling vertically with a velocity of 88 ft. per second. Find the speed of the train.

9. A ship sails N.E. at 10 miles an hour, and to a passenger on board the wind appears to blow from the north with a velocity of $10\sqrt{2}$ miles an hour. Find the true velocity of the wind.

10. Two trains whose lengths respectively were 130 and 110 ft., moving in opposite directions on parallel rails, were observed to be 4 secs. in completely passing each other, the velocity of the longest train being double that of the other; find at what rate per hour each train is moving.

11. Two particles move with velocities v and $2v$ respectively in opposite directions, in the circumference of a circle. In what positions is their relative velocity greatest and least respectively, and what values has it then?

12. A man walking 4 miles an hour is struck by the rain vertically. When he runs at eight miles an hour, the rain drives in his face at an angle of 45°. Find the direction in which the rain is driven by the wind, and its velocity.

13. Two trains start from the same spot, one due north at 60 miles an hour, another due west at 25 miles an hour: find their relative velocity.

14. To a man walking due west, the wind appears to come from the north-west; but when he walks due east at the same rate, it appears to come from the north-east. Compare the velocities of the wind and the man.

15. An ironclad, steaming due north at 16 miles an hour, rams another steaming due east at 12 miles an hour. Find the direction of the blow as felt by the struck vessel.

16. A ship is steaming in a direction 30° south of east; another, 12 miles south of the first, is moving due north at the same speed Find their shortest distance apart.

17. A ship steams due north at 6 miles an hour, and a man runs across her deck so that his actual velocity is 6 miles an hour, and his actual direction in space is due west. Find his velocity relative to the ship, and the point of the compass he faces whilst running.

18. A man, running with velocity v, passes through rain falling vertically with velocity $2v$. He holds an open tube in such a position that a drop entering it passes through it along its axis: at what elevation does he hold it?

19. A ship is steaming due east, and the apparent direction of the wind, as shown by a vane on the mast, is from the south. The wind is known to be blowing from a point 30° west of south; prove that its velocity is twice that of the ship.

20. One ship sailing east with a speed of 15 miles an hour passes a certain point at noon; and a second ship sailing north at the same speed passes the same point at 1.30 p.m. At what time are they closest together, and what is then the distance between them?

21. Two particles P and Q start simultaneously from A, one sliding down the plane AB at an angle a to the horizon, and the other falling freely: prove that their relative vertical acceleration is $g \cos^2 a$. Prove also that the line PQ is always perpendicular to AB.

22. A light inextensible string, passing over a smooth fixed pulley, is attached at one end to a mass of 8 lbs., and at the other to a second smooth pulley over which passes another similar string carrying masses of 3 lbs. and 5 lbs. at its ends. Find the tensions of the strings, the masses of the pulleys being negligible.

23. A light inextensible string, passing over a smooth fixed pulley, is attached at one end to a mass m, and at the other to a second smooth pulley of negligible mass, over which passes another similar string carrying masses of 2 lbs. and 4 lbs. at its extremities; find the value of m when it descends with uniform velocity.

24. A mass of 8 lbs. lying on a smooth horizontal table is attached to an inextensible string which, passing over the edge of the table at right angles, carries a smooth weightless pulley, over which hangs another inextensible string carrying masses of 3 lbs. and 5 lbs. at its ends: find the acceleration of the body on the table.

Harder Pulley Problems.

25. Equal weights each of mass m are attached to the end of a fine string which passes over two fixed smooth pulleys and under a smooth moveable pulley, also of mass m; the pulleys being so arranged that all the parts of the string which are not in contact with the pulleys are vertical. Find the upward acceleration of the moveable pulley and the tension of the string.

26. In the system of pulleys described in the preceding question, find the acceleration of the moveable pulley and the tension of the string when masses of 5 and 3 lbs. are hung at the ends of the string, and 12 lbs. is the mass of the moveable pulley.

27. An ordinary block and tackle (the system of pulleys in which there is one continuous string) has two pulleys in the lower block and two in the upper block. If a mass of m lbs. is attached to the free end of the string, and the mass of the lower block and attached weight is also m lbs., find the tension of the string and the acceleration of the lower block.

28. A light string fastened at one end passes under a smooth pulley of mass 12 lbs., and over a fixed smooth pulley, and carries a mass of 5 lbs. at its free end. The different parts of the string not in contact with the pulleys being vertical, find the tension of the string and the acceleration of the moveable pulley.

29. Solve the preceding problem when m_1 is the mass of the moveable pulley, and m_2 that of the attached weight.

What will be the acceleration of m_1 if its mass is *very small* compared with that of m_2?

What will be the acceleration of m_2 if its mass is *very small* compared with that of m_1? Give your reasons in each case.

30. A light string passing over a smooth pulley has a spring balance weighing 4 oz. attached to one end and a weight of 12 oz. to the other. A weight of 4 oz. is placed in the balance and the system is allowed to move freely. What weight is indicated by the balance?

Trail of Smoke Problems.

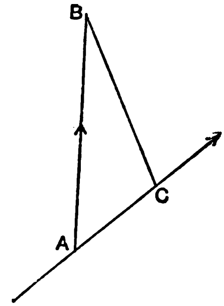
89. In such problems as the following, it is assumed that when the smoke leaves the funnel of the engine **it immediately loses the velocity of the engine** by reason of the atmospheric resistance, and takes on the velocity of the wind.

Let AB be the path of an engine travelling uniformly, and BC its trail of smoke at any instant.

Suppose the particles of smoke now at C left the engine at A. Then AC is the direction of the wind and whilst the engine has travelled from A to B, the wind has carried these particles of smoke from A to C.

Hence if u and v be the velocities of the train and wind, and t the time the train takes to travel from A to B,

$$AB = ut, \text{ and } AC = vt.$$



EXAMPLES. XI *b*.

1. A train is travelling northwards at 40 miles an hour, and the wind is blowing from the south-west at 20 miles an hour. Shew in a diagram, the direction of the trail of smoke of the engine. Give explanations.

2. A train runs in a north-easterly direction at 20 miles an hour, and the wind is blowing from the south. If the trail of smoke makes an angle of 20° with the direction of the train, find the velocity of the wind, and verify your result graphically.

3. The trail of smoke from a steamer running east at 6 miles an hour is observed to extend in a direction 70° north of west, whilst that from another running west at the same speed is observed to be 40° north of east. Find graphically the velocity and direction of the wind.

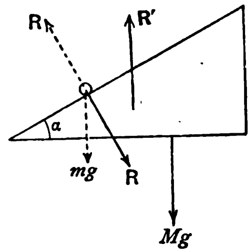
4. From a train travelling north with velocity v, the trail of smoke points 60° east of south, whilst from a train running east at the same speed the trail points north. Find the direction of the wind and verify your result graphically.

5. A train travels due north at a uniform speed and its trail of smoke points in a direction 40° east of south when the wind blows in a direction 50° east of north. In what direction will the trail of smoke point when the velocity of the wind is doubled?

Find the direction in which the trail of smoke will point if the velocity of the wind is trebled instead of doubled.

90. *A wedge of mass M and angle α is free to move on a smooth horizontal table; a particle of mass m slides down its inclined face: find the acceleration of the wedge along the table.*

Let f be the acceleration of the wedge.



The forces acting on the wedge are

(1) its own weight vertically downwards;

(2) the resistance of the table R';

(3) the pressure (R) of the particle on the inclined face.

These forces are shewn in continuous lines.

The forces (shewn in dotted lines) acting on the particle are

(1) its weight mg vertically downwards;

(2) the pressure of the inclined face.

Considering the motion of the wedge,

$$R \sin \alpha = Mf \quad (P = mf) \text{ ....(1).}$$

Again, since the particle remains in contact with the wedge, its acceleration *at right angles* to the inclined face must be equal to that of the wedge in the same direction, i.e. equal to $f \sin \alpha$.

Also the resultant force on the particle at right angles to the inclined face

$$= mg \cos \alpha - R,$$

$$\therefore\ mg \cos \alpha - R = mf \sin \alpha, \qquad (P = mf)$$

i.e. from equation (1)

$$mg \sin \alpha \cos \alpha - Mf = mf \sin^2 \alpha,$$

$$\therefore\ f = \frac{mg \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}.$$

EXAMPLES. XI *c*.

1. A smooth wedge whose slant side is 13 ft. and height 5 ft. is placed on a horizontal table, and a body whose mass is 1 lb. is placed on the wedge. If the mass of the wedge is 12 lbs., shew that it moves along the table with an acceleration $\frac{60g}{2053}$.

2. In the above problem prove that the pressure on the slant side of the wedge during motion is $\frac{1872}{2053}$ lbs. wt., and that the vertical acceleration of the 1 lb. mass is $\frac{325g}{2053}$.

How long will the 1 lb. mass take to slide down the whole slant face?

[Leave your answer in surdic form.]

3. A smooth inclined plane of mass 6 lbs. and elevation 45° is free to move on a horizontal table. A mass of 3 lbs. slides down its slant face. Find the horizontal force which acting on the inclined plane will prevent its motion.

4. If, in the previous problem, the inclined plane is allowed to move freely, prove that the acceleration of the 3 lb. mass relative to the plane is $\frac{\sqrt{10}g}{5}$, inclined at an angle $\tan^{-1}\frac{1}{3}$ to the vertical.

5. A smooth inclined plane of mass 10 lbs. and elevation 45° is free to move along a horizontal plane. A mass of 5 lbs. slides down its slant face. How far will the plane move in the first second of its motion if the 5 lb. mass remains on it during that time?

If the 5 lb. mass leaves the plane after 2 seconds, how far will the plane move in the next 3 seconds?

6. A smooth inclined plane of mass m and elevation a is free to move along a horizontal table. A steady vertical force P acts upon its slant face: prove that the plane moves along the table with an acceleration $\frac{P}{m}\sin a \cos a$.

7. A smooth inclined plane of elevation a is constrained to move with a uniform velocity in a horizontal direction, whilst a particle slides down its slant face. Find the acceleration of the particle.

ANGULAR VELOCITY.

91. DEF. *If in any plane, O be a fixed point, P any other point, and OA a fixed straight line, then the rate at which the angle AOP changes is called the* **angular velocity** *of P about the point O.*

Angular velocity is measured in radians.

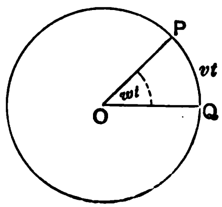
Thus the **unit of angular velocity** is that uniform velocity with which a point describes the unit angle (a radian) in the unit of time.

Uniform angular velocity is measured by the number of unit angles (radians) described in the unit of time.

Variable angular velocity is measured at any instant by the number of unit angles which would be described in the unit of time, if during that unit the angular velocity remained the same as at the instant under consideration.

92. *A point describes a circle with uniform velocity; to express this linear velocity (v) in terms of the angular velocity (ω) of the point about the centre of the circle.*

Let the point move from P to Q on the circumference of the circle, centre O, radius r, in time t; so that arc $PQ = vt$. $(s = vt)$



Then since ω radians is the angular velocity of the point about O, ωt radians is the angle described by OP in time t.
$\therefore \angle POQ = \omega t$ radians.

But the number of radians in

$$\angle POQ = \frac{\text{arc } PQ}{OQ}.$$

$$\therefore \omega t = \frac{vt}{r},$$

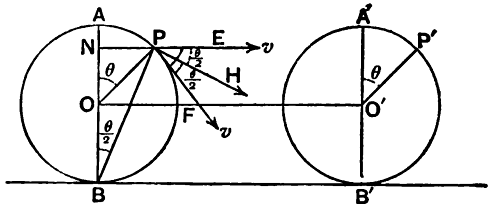
i.e. $$\mathbf{v = r\omega}.$$

93. Ex. i. *A point is moving in a circle of radius* 10 *feet with a uniform velocity of* 20 *feet per second: find its angular velocity.*

Let ω radians per sec. be its angular velocity.

Then since $$v=r\omega,$$
$$20=10\omega,$$
i.e. $$\omega=2 \text{ radians per second.}$$

Ex. ii. *A wheel rolls uniformly along the ground without slipping, its centre describing a straight line; to find the velocity of any point on its rim.*



Let v be the velocity of the centre of the wheel, radius r; and suppose the wheel to roll through one circumference, from the position APB to the position $A'P'B'$. Then $OO'=BB'=$circumference of the wheel, since there is no slipping; i.e. whilst the centre O has described a circumference, any point P on the wheel has, in the same time, also described, relative to the centre, one circumference.

Hence the velocity of any point on the rim, relative to the centre, is equal to that of the centre itself.

Thus any point P on the rim has *two equal velocities*, one in the direction of motion of the centre, and the other along the tangent to the circle at the point P.

Draw NPE parallel to OO', PF the tangent at P, PH bisecting the angle EPF, and let $\angle PON=\theta$.

$$\angle EPF=\frac{\pi}{2}-\angle OPN=\theta.$$

$$\therefore \angle EPH=\angle FPH=\frac{\theta}{2}.$$

Therefore the resultant velocity of the point $P=2v\cos\frac{\theta}{2}$ along PH.

Cor. 1. At the highest point of the wheel $\theta=0$ and the velocity of the point A is therefore $2v$ in the direction of motion of the centre. At the lowest point B, the two velocities, parallel to OO' and relative to the centre, are equal and opposite; the point B is therefore ***instantaneously at rest.*** (At this point $\theta=\pi$.)

COR. 2. $\angle BPH = \angle OPF =$ a right angle, i.e. P is moving at right angles to BP.

Hence its angular velocity about the point B

$$=\frac{2v\cos\frac{\theta}{2}}{BP}=\frac{2v\cos\frac{\theta}{2}}{2r\cos\frac{\theta}{2}}=\frac{v}{r},$$

$=$ angular velocity of P about the centre.

At the instant, the point P is moving at right angles to BP, and hence B is sometimes called the *instantaneous centre of rotation* of the point P.

Ex. iii. *A particle, at the point A, has a velocity u making an angle θ with the line OA: to find the angular velocity of the particle at the instant about the point O.*

Let ω be the angular velocity required.

The velocity u is equivalent to

$$u\cos\theta \text{ along } AO,$$

and $$u\sin\theta \text{ perpendicular to } AO.$$

The component $u\cos\theta$ has no angular velocity about O. $\therefore \omega =$ the angular velocity of the component $u\sin\theta$ about O

$$=\frac{u\sin\theta}{AO}. \qquad \text{(Art. 92.)}$$

EXAMPLES. XI *d*.

Angular Velocity.

[Take $\pi = \frac{22}{7}$.]

1. A line turns through three-quarters of a right angle in one second; find its angular velocity.

2. A point describes a circle of radius 5 feet with the unit of angular velocity; find its linear velocity.

3. A wheel makes 50 revolutions per minute about its centre: find the angular velocity of any point on the wheel about its centre.

4. A point moves on the circumference of a circle of radius 3 feet with a velocity of 12 feet per sec.: find its angular velocity, and the angle (in degrees) its radius turns through in half a second.

5. Compare the angular velocities of the hour-hand, the minute-hand, and the second-hand of a watch.

6. From a train moving with velocity v, a carriage, on a road parallel to the line at a distance d from it, is observed to move so as to appear always in a line with a more distant fixed object whose least distance from the railway is D. Find the velocity of the carriage.

7. The wheel of a carriage is 4 feet in diameter and it makes $3\frac{1}{2}$ revolutions per second; find the angular velocity and the linear velocity of any point in its circumference relative to the centre.

8. A point describes a circle with uniform velocity; show that its angular velocity about any point on the circumference of the circle is constant.

9. A string has one end attached to a corner of an equilateral triangle placed on a smooth horizontal table, and is wound round the triangle carrying a particle at its other end; the particle is projected with velocity u at right angles to the side (length a) of the triangle: if the length of the string be $3a$, find the time that the string takes to unwrap itself from the triangle, assuming that the velocity of the particle remains the same throughout the motion.

10. What is the angular velocity of the earth about its axis?

11. The velocity of the extremity of the minute-hand of a clock is 30 times that of the extremity of the hour-hand, which is 4 inches long: find the length of the minute-hand.

12. What is the angular velocity of the earth about its axis if the unit of angular velocity is that of the minute-hand of a watch?

13. A point moves uniformly along a straight line: show that its angular velocity about any point varies inversely as the square of its distance from that point.

14. If at the equator a straight hollow tube were thrust vertically down to the centre of the earth, and a heavy particle were dropped from a point on the axis of the tube, it would soon strike one side: find which, and give a reason for your answer.

15. An engine travels at 45 miles an hour: find the velocities at any instant of the points on its driving wheel (6 ft. diameter) at a height 3 ft. from the ground.

16. In the previous example find at any instant the velocity of (1) the highest point of the driving wheel, (2) the point in contact with the ground.

17. Two points describe the same circle in such a manner that the line joining them always passes through a fixed point: show that at any instant their velocities are proportional to their distances from that point.

18. The highest point of a wheel at a distance of 4 feet from the point in contact with the ground is moving with an instantaneous velocity of 40 ft. per second: what is its angular velocity about its instantaneous centre, and what about the centre of the wheel?

19. If a line turn round its extremity O, which is a fixed point within a given circle, with uniform angular velocity, show that the velocity with which its intersection P with the circle travels along the circumference varies as PH, which is a line drawn from P through the centre, to meet OH a perpendicular to OP.

20. What is the angular velocity of the earth if the unit of angular velocity is that of the hour-hand of a watch?

What is it in radians per hour?

21. A ring of radius a rolls on the concave side of the circumference of a fixed ring of radius ma, the point of contact having the velocity v. Shew that the angular velocity of the moving ring is

$$\frac{m-1}{m} \cdot \frac{v}{a}.$$

If $m=2$, shew that every point on the moving ring describes a diameter of the fixed ring.

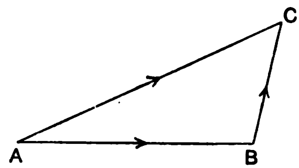
22. If a railway carriage is moving at the rate of 30 miles an hour, and the radius of one of its wheels is 2 ft., what is the angular velocity of the wheel if there be no slipping?

23. What is the relative angular velocity of the hour and minute-hands of a clock?

24. A rod OA revolves about O in a plane with a uniform angular velocity of $\frac{\pi}{9}$ radians per second. As the rod revolves a particle slides from A, 40 inches from O, along the rod towards O with a uniform velocity of 5 inches per sec. Draw a representation of the path of the particle for the first 8 secs. of its motion.

25. Prove that if the velocity of a particle is resolved into several components in one plane, its angular velocity about any fixed point in the plane is the sum of the angular velocities due to the several components.

94. Change of Velocity. We shall consider here the case of a moving point whose velocity is changing in *direction* as well as in magnitude.



Let AB represent the velocity of the point at any time, and AC its velocity after any further time t. Join BC.

Then AC represents the resultant of the two velocities represented by AB and BC (Triangle of Velocities); or the velocity AC is equivalent to the original velocity AB, compounded with the velocity BC.

Thus BC represents the change of velocity in the time t. Moreover, if the rate of change of velocity is constant in magnitude and direction, $\frac{BC}{t}$ = the change of velocity in unit time, or the acceleration.

EXAMPLES. XI *e*.

1. If ABC is a straight line and AB represents the velocity of a particle and BC represents its uniform acceleration, what does AC represent?

2. Two straight lines AB, BC represent the velocity of a particle at a certain time, and its uniform acceleration in the direction of its velocity. Find the velocity of the particle in 3 seconds.

3. A body has a velocity of 20 ft. per sec. and 3 secs. afterwards is found to have an equal velocity making an angle of 40° with its first direction. Find the change of velocity, and also its average acceleration in magnitude and direction.

4. In 4 secs. the velocity of a body changes from 25 ft. per sec. to 15 ft. per sec. making an angle of 50° with its first direction, and the body is known to be subject to a uniform acceleration. Find the magnitude of the acceleration and the direction and magnitude of the velocity of the body after two seconds. (Graphical.)

5. Straight lines OA, OB, OC, OD represent in magnitude and direction the velocities of a particle at intervals of one second. What do you know about the figure $OABCD$... if the particle has a uniform acceleration in a straight line? Give reasons.

6. A particle has a velocity of 40 ft. per sec. due north, and 6 seconds afterwards is found to have a velocity of 60 ft. per sec. due east. Assuming it to be subject to a constant acceleration in a straight line, find graphically the magnitude and direction of its velocity after 1, 2, 3, 4, and 5 seconds.

7. A particle travels with uniform velocity, v, round the perimeter of a regular polygon. If r is the radius of the circum-circle of the polygon, prove that at each angle of the polygon, the particle receives an additional velocity towards the centre of the circle. Also prove that the additional velocity per unit time $=\frac{v^2}{r}$.

8. Draw BAC an angle of 120°. A particle moves from B to A with a velocity of 15 ft. per second; when it comes to A a new velocity is impressed on it and, in consequence, it next moves from A to C with a velocity of 15 ft. per second; find, by construction or calculation, the magnitude and direction of the impressed velocity.

9. A body moving along a straight line AB with a velocity of 6 ft. per second, has, on arriving at B, an additional velocity impressed on it, and then moves along BC with a velocity of 10 ft. per second, where the angle $ABC = \cos^{-1}\left(-\frac{3}{5}\right)$. Find the magnitude and direction of the impressed velocity.

CHAPTER XII.

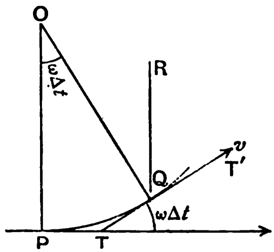
UNIFORM CIRCULAR MOTION.

95. *A particle describes a circle with uniform velocity, to shew*

(1) *that it is subject to an acceleration,*

(2) *that this acceleration is towards the centre of the circle,*

and (3) *to find the magnitude* $\left(\frac{v^2}{r}\right)$ *of that acceleration.*



(1) By the first Law of Motion, there must be some *force* acting upon the particle, or it would move uniformly in a straight line; and therefore by the second Law of Motion, the particle is subject to an *acceleration.*

(2) This acceleration can have no resolved part in the direction of motion; for if it had, this resolved part would produce a *change* of speed. Therefore the acceleration is at right angles to the direction of motion, i.e. towards the centre of the circle.

(3) To find the magnitude of this acceleration.

Let r be the radius of the circle whose centre is O, and let the particle move with velocity v.

Also let ω be its angular velocity, so that $v = r\omega$.

Take two points P and Q on the path very close to one another, so that $\angle POQ$ is very small, and let the particle move from P to Q in time Δt, so that $\angle POQ = \omega\Delta t$.

Draw the tangents PT, QT' to the circle at P and Q, and draw QR parallel to PO.

When the particle is at P its velocity in the direction PO is zero.

When it arrives at Q its velocity in the same direction is

$$v \cos RQT' = v \sin(\omega\Delta t).$$

$\therefore$ in time Δt a velocity $v \sin(\omega\Delta t)$ has been generated in the direction PO.

$\therefore$ the acceleration in that direction

$$= \text{the limiting value of } \frac{v \sin(\omega\Delta t)}{\Delta t} \qquad \left(\frac{\Delta v}{\Delta t}\right),$$

when the angle $\omega\Delta t$, and the time Δt are indefinitely small.

But when the angle is very small,

$$\sin(\omega\Delta t) = \omega\Delta t \qquad (\sin\theta = \theta).$$

$$\therefore \text{ the acceleration} = \text{the limiting value of } \frac{v\omega\Delta t}{\Delta t}$$

$$= v\omega = v \times \frac{v}{r}$$

$$= \frac{v^2}{r}.$$

COR. 1. If m be the mass of the particle,

$$\text{the } \textit{force} \text{ towards the centre} = \frac{mv^2}{r}.$$

COR. 2. The acceleration $\frac{v^2}{r}$ may be written in the form

$$r\omega^2, \text{ for } v = r\omega.$$

96. Thus if a particle of mass m lbs. move on a smooth horizontal table with velocity v ft. per sec. being attached to a fixed point on the table by a string of length r, the tension of the string is the force which prevents the particle from leaving the circle.

Therefore the tension of the string is $\frac{mv^2}{r}$ poundals.

The force exerted upon the particle by the string is *towards* the centre and is called a *centripetal force*.

The force exerted upon the string by the particle is directed from the centre and is called a *centrifugal force.*

By the third Law of Motion these two forces are equal and opposite.

The student must be very careful to notice that the force acting on the particle is towards the centre and therefore centripetal. The centrifugal force does not act on the particle.

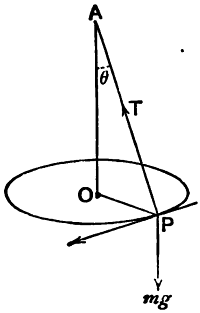
97. Conical Pendulum. *A particle P, of mass m, is tied by a string of length l to a fixed point A, and is made to describe a horizontal circle with uniform velocity, such that it makes n complete revolutions per second, the string generating a cone whose axis is the vertical line through A: to find (T) the tension of the string, and its inclination (θ) to the vertical.*

Let O be the centre of the circle described by the particle, r its radius, v the velocity of the particle.

$$r = l \sin\theta \text{ since } \angle AOP \text{ is a right angle,}$$

$$\therefore\ v = 2n\pi r = 2n\pi l \sin\theta \ \ldots\ldots\ldots\ldots\ldots\ (1).$$

The only forces acting upon the particle are the tension T of the string along PA, and the weight mg of the particle.



The particle has no vertical motion.

Therefore resolving in that direction

$$T\cos\theta = mg \ \ldots\ldots\ldots\ldots\ (2).$$

Also since the particle describes a circle, radius r, with uniform velocity v, the resultant force towards the centre

$$= m\frac{v^2}{r}.$$

$$\therefore\ T\sin\theta = \frac{mv^2}{r} = \frac{m \,.\, 4n^2\pi^2 l^2 \sin^2\theta}{l\sin\theta} \text{ from (1),}$$

$$= m \,.\, 4n^2\pi^2 l \sin\theta,$$

$$\therefore\ T = m \,.\, 4n^2\pi^2 l \text{ poundals} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (3).$$

Also from equations (2) and (3) by division

$$\cos\theta = \frac{g}{4n^2\pi^2 l} \dots\dots\dots\dots\dots\dots (4),$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{g}{4n^2\pi^2 l}\right).$$

We have thus found both T and θ.

COR. Since the particle makes n revolutions in one second, the time of one revolution

$$= \frac{1}{n} \text{ secs.}$$

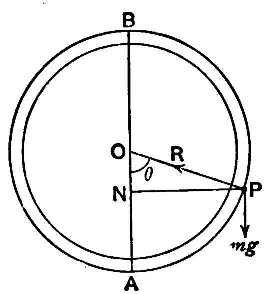
$$= \sqrt{\frac{4\pi^2 l \cos\theta}{g}} \text{ from equation (4)}$$

$$= 2\pi\sqrt{\frac{OA}{g}},$$

$$\propto \sqrt{OA}.$$

Hence the time of revolution of a particle moving in a conical pendulum varies as the square root of the depth of the particle below its point of suspension.

98. *A smooth hollow tube, bent into a circular form, rotates with uniform angular velocity (ω) about a vertical diameter; to show that a heavy particle placed in the tube will remain at rest relative to the tube in one particular position.*



Let APB be the tube, AB the vertical diameter of rotation, O the centre of the tube.

Suppose that when the particle, mass m, is placed at P, it rotates with the tube, remaining at rest relative to the tube. Let $\angle AOP = \theta$, r be the radius OP. Draw PN horizontally to meet the axis AB at N.

The only forces acting on the particle are

(1) its weight mg vertically downwards;

(2) the pressure of the tube.

Considering the position of the particle at P in the plane of the paper, we see that the pressure of the tube on the particle can have no component at right angles to the plane of the paper, for in that case the particle's velocity would change.

The pressure of the tube must therefore act in the plane of the paper and therefore along the normal PO at P, the tube being smooth.

Let R poundals be this pressure.

By hypothesis the particle has no vertical motion, therefore resolving in that direction

$$R \cos\theta = mg \quad\text{.......................(1).}$$

Also the particle (by hypothesis) describes a horizontal circle radius PN, with uniform angular velocity,

$$\therefore\ R\sin\theta = m\,.\,\omega^2 PN \qquad \text{(Art. 95, Cor. 2)}$$

$$= m\,.\,\omega^2 r\sin\theta.$$

Therefore either (1) $R = m\omega^2 r$ or (2) $\sin\theta = 0$.

(1) If $R = mw^2r$, we have from equation (i)

$$\cos\theta = \frac{g}{\omega^2 r}.$$

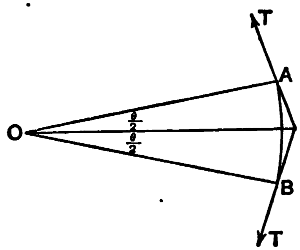
Hence if $g < \omega^2 r$ we can find an angle $\cos^{-1}\left(\frac{g}{\omega^2 r}\right)$, and the particle will rest relative to the tube at the point P, where

$$\angle POA = \cos^{-1}\left(\frac{g}{\omega^2 r}\right).$$

(2) If $\sin\theta = 0$ or if $g > \omega^2 r$ the particle will rest at the lowest point of the tube.

99. *A heavy uniform ring revolves in a horizontal plane with uniform angular velocity about its centre, to find the stress at any point of the ring.*

Let m be the mass of the ring, r its radius, so that $\frac{m}{2\pi r}$ = the mass of unit length of the ring.



Consider a small portion, AB, of the ring, which subtends an angle θ at the centre O; and let T be the stress at each end of the portion along the tangents at A and B. The mass of AB is

$$\frac{m \, . \, AB}{2\pi r}.$$

Since AB describes a circle with uniform angular velocity ω, the resultant force towards the centre

$$= \frac{m \, . \, AB}{2\pi r}(r\omega^2).$$

$$\therefore \; 2T \sin\frac{\theta}{2} = \frac{m \, . \, AB \, . \, r\omega^2}{2\pi r}$$

$$= \frac{m \, . \, r\theta\omega^2}{2\pi}.$$

But θ is a *small* angle, therefore $\sin\frac{\theta}{2} = \frac{\theta}{2}$.

Hence $$T \, . \, \theta = \frac{mr\theta\omega^2}{2\pi},$$

i.e. $$T = \frac{mr\omega^2}{2\pi}.$$

In the above, the mass of unit length $= \frac{m}{2\pi r} = m'$ suppose.

Then $$T = m' \, . \, r^2\omega^2$$
$$= m' \, . \, v^2$$

where v is the linear velocity of the unit mass.

Hence the stress at any point of a revolving wheel varies as the **square of the velocity** of the point: and this offers an explanation of the fact that wheels running at a great speed sometimes fly to pieces.

Ex. i. *A carriage whose mass is 4 tons, moving at the rate of 30 miles an hour, goes round a curve whose radius is a quarter of a mile: find the horizontal outward pressure on the rails.*

The velocity of the train $= 30$ miles an hour

$= 44$ feet per sec.

The mass of the carriage $= 4 . 2240$ lbs.

The radius of the circle described $= 1320$ feet.

Therefore the horizontal outward pressure on the rails

$=$ the force which keeps the carriage in the circle

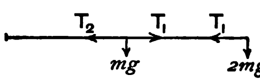
$$= \frac{4 . 2240 . (44)^2}{1320} \text{ poundals} \qquad \left(\frac{mv^2}{r}\right)$$

$$= \frac{4 . 2240 . (44)^2}{1320 . 32} \text{ lbs. wt.}$$

$$= 410\tfrac{2}{3} \text{ lbs. wt.}$$

Ex. ii. *A particle of mass 2m is fastened to the end of a string of length 2r, and a mass m to its middle point, the other end being attached to a point on a smooth horizontal table. The masses are then projected so that they revolve in circles on the table with uniform velocities, and such that the portions of the string remain in a straight line. Find the ratios of the tensions of the two portions of the string.*

Let v be the velocity of the smaller particle. Then since the larger describes a circle of twice the radius (and therefore twice the circumference) of the smaller circle in the same time, its velocity $= 2v$.



Let T_1 be the tension of the part of the string between the masses, T_2 that of the other part.

The resultant force acting on the smaller particle towards the centre of the circle $= T_2 - T_1$.

$$\therefore\ T_2 - T_1 = \frac{mv^2}{r} \quad \text{.........................(1).}$$

Also since T_1 is the only force acting on the larger particle towards the centre of its circle,

$$T_1 = \frac{2m\,(2v)^2}{2r} = \frac{4mv^2}{r} \quad \text{.....................(2).}$$

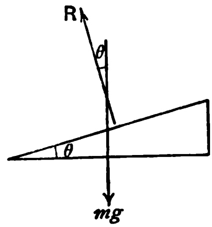
Adding equations (1) and (2)

$$T_2 = \frac{5mv^2}{r},$$

$$\therefore\ T_1 : T_2 :: 4 : 5.$$

Ex. iii. *A train is travelling at the rate of* 60 *miles an hour on a curve whose radius is* 880 *yards. When the distance between the rails is* 5 *ft., find to two decimal places how many inches the outer rail must be raised above the inner, so that there may be no lateral pressure on the rails.*

Let m be the mass of the train, R the resultant normal pressure on the rails, inclined at an angle θ to the vertical, so that θ is the inclination of the floor of the carriage to the horizon.



There is no vertical motion, therefore resolving in that direction

$$R\cos\theta = mg \text{ ....(1).}$$

Since the carriage describes a circle of radius 2640 ft. with a velocity of 88 ft. per sec. ($=60$ miles an hour), the resultant force towards the centre of the circle

$$= \frac{m \cdot 88^2}{2640}, \qquad \left(\frac{mv^2}{r}\right)$$

$$\therefore\ R\sin\theta = \frac{m \cdot 88^2}{2640} \text{ ....(2).}$$

From equations (1) and (2) by division,

$$\tan\theta = \frac{88^2}{2640 \cdot 32} = \frac{11}{120} = {\cdot}0917 \text{ approx.}$$

$$\therefore\ \theta = 5^\circ\ 14' \text{ (from Tables).}$$

And the required height $= 60 \sin\theta$

$= 60 \sin 5^\circ\ 14'$

$= 60 \times {\cdot}0912$

$= 5{\cdot}47$ inches.

EXAMPLES. XII.

[$\pi = \frac{22}{7}$ unless otherwise inferred in the question.]

1. A mass of 4 lbs. is tied to the end of a string 3 feet long, and made to revolve 7 times in 22 seconds in a circle; find the tension of the string.

2. A point describes the circumference of a circle whose radius is 4 feet with a velocity of 8 feet per sec.; find its acceleration.

3. A railway carriage, whose mass is 3 tons, moves at the rate of 20 miles an hour round a curve whose radius is 110 yards; find the outward pressure against the rails.

4. A railway carriage, moving at the rate of 30 miles an hour, goes round a curve whose radius is 880 feet. A mass of 6 lbs. hangs by a string from the roof; what horizontal force will keep the string vertical while the carriage is rounding the curve?

5. How many times a minute must a mass of 5 lbs. revolve horizontally at the end of a string 7 ft. long, so that the tension of the string may be equal to the weight of $17\frac{1}{2}$ lbs.?

6. A mass of 14 lbs. revolves in a horizontal plane at the end of a string 7 ft. long; find the tension of the string when 2 secs. is the time of one revolution.

7. A mass of 5 lbs. is suspended by a string from the roof of a railway carriage moving at 15 miles an hour round a curve whose radius is 220 feet; find the inclination of the string to the vertical.

8. An engine runs round a curve of radius r with velocity v; at what angle must the 'way' be inclined to the horizon so that there may be no lateral pressure on the rails?

9. A train travels round a curve whose radius is 1210 feet with a speed of 30 miles an hour. If the distance between the rails be 5 ft., how much must the outer rail be raised above the inner so that there may be no lateral pressure on the rails?

10. A railway carriage is moving at the rate of 45 miles an hour on a curve of radius 1320 yards, and a mass is suspended from the roof

of the carriage by means of a string 4 ft. long; show that the mass will move through 1·65 inches approximately from the vertical.

11. A train travels round a curve whose radius is 121 feet at a speed of 15 miles an hour; show that the outer rail must be raised 7·44 inches approximately above the inner if there is no lateral pressure on the rails, the distance between the rails being 5 feet.

12. A mass of 4 lbs. revolves in a horizontal circle at the end of a string 4 feet long. The string can just bear a strain of 8 lbs. wt.; find the greatest number of complete revolutions a minute the mass can make.

13. A skater, whose weight is 12 stone, cuts on the outside edge a circle of 12 ft. diameter with a uniform velocity of 8 ft. per sec.; find the magnitude and direction of his pressure on the ice.

14. A mass of 5 lbs., attached to a string $4\sqrt{3}$ ft. long, revolves as a conical pendulum with a speed of 8 ft. per second; find the tension of the string and its inclination to the vertical.

15. A string 3 ft. long has a mass of 4 lbs. attached to one end and revolves, as a conical pendulum, 7 times in 11 seconds; find the tension of the string and its inclination to the vertical.

16. A particle of mass m on a smooth horizontal table is fastened to one end of a fine string which passes through a small hole in the table, and supports at its other end a particle of mass $2m$, the particle m being held at a distance c from the hole. Find the velocity with which m must be projected horizontally so as to describe a circle of radius c.

17. If different points be describing different circles with uniform velocities, and with accelerations proportional to the radii, the periodic times will be the same.

18. A skater describes a circle 100 ft. in radius with a velocity of 18 ft. per sec. What is the cotangent of his inclination to the ice?

19. Two masses m_1 and m_2 are placed on a smooth table, and connected together by a string passing through a small fixed ring on the table. If they are projected with velocities v_1 and v_2 at right angles to the string, find the ratio in which the string must be divided in order that both masses may describe circles round the ring as centre.

20. A smooth straight tube, inclined at an angle α to the vertical, revolves round a vertical axis with angular velocity ω; at what point in the tube will a particle rest in equilibrium relative to the tube?

21. A mass of 10 lbs. is fastened by a string which passes through a hole in a smooth horizontal table to a mass of 1 lb., which hangs vertically; the first weight is revolving on the table about the hole as centre; how many revolutions are there per minute if the horizontal portion of the string is 15 inches long?

22. If a conical pendulum be 10 feet long, the half angle of the cone 30°, and the mass of the bob 12 lbs., find the tension of the thread and the time of one revolution.

23. A thin hollow cylinder 30 inches in circumference, and closed at the lower end, is revolving about its axis, which is vertical, once in every second. A rough flexible chain, made into a ring of the same radius, is dropped into the cylinder. If the mass of the chain is 3·75 ozs., find the ultimate pressure on the cylinder per foot.

24. Show that the vertical depth of the centre of gravity of a conical pendulum below its point of support depends only on its speed of rotation and the intensity of gravity, and is independent of its length.

25. A wet open umbrella is held with the handle upright and made to rotate round that handle at the rate of 14 revolutions in 33 secs. If the rim of the umbrella be a circle of one yard diameter, prove that the force required to keep a drop of water weighing ·01 oz. attached to the rim is ·0105 oz. wt. making an angle $\tan^{-1}\frac{1}{3}$ with the vertical.

26. A string passing through a small hole in a smooth horizontal table has a small sphere of mass m attached to each end of it; the upper sphere revolves in a circle on the table, when suddenly it strikes an obstacle and loses half its velocity; find what diminution must be made in the weight of the lower sphere in order that the upper sphere may continue moving in a circle.

27. A mass m is fastened by a string 5 feet long to a point 3 feet above a smooth table; if the particle be made to revolve as a conical pendulum on a table with a velocity of 4 feet per second, find the pressure on the table.

28. A point P describes a circle with a given uniform velocity v. If Q be the projection of P on a given fixed diameter, show that the velocity of Q varies as PQ, and that its acceleration varies as QO, where O is the centre.

29. If a train is running round a curve of radius r with velocity v, show that the weight of a carriage is divided between the outer and inner rails in the ratio of $gra+v^2h$ to $gra-v^2h$, when h is the perpendicular distance of the C.G. of the carriage from the plane of the rails, and $2a$ the distance between the rails.

30. A particle weighing $\frac{1}{4}$ oz. rests on a horizontal disc, and is attached by two strings 4 ft. long to the ends of a diameter. If the disc be made to revolve 100 times a minute about a vertical axis through its centre, find the tension of each string.

31. A particle of mass m lbs. is attached by a light string 2 feet long to a fixed point A on a smooth horizontal table, and a particle of mass $2m$ lbs. is attached to the former particle by a light string 1 ft.

long. The system revolves uniformly on the table making one revolution per sec. round A, the strings being stretched and in the same straight line. Find the tension of each string in pounds' weight.

32. A stone of 6 lbs. wt. is whirled round uniformly in a horizontal plane by a string two yards long, one end of the string being fixed; find the time of a revolution when the tension of the string is equal to the weight of $\dfrac{36\pi^2}{g}$ lbs.

33. Two particles of the same weight are fastened respectively to the middle point and one extremity of a weightless string, and laid upon a smooth table, the other extremity of the string being fastened to a point in that table. If the string be pulled straight and the particles so projected that they remain in a straight line, prove that the tensions in the two portions of the string will be as 3 : 2.

34. A uniform ring of radius $1\frac{3}{22}$ ft. and weighing 4 lbs. a foot revolves horizontally about its centre 100 times a second. Show that it must be able to bear a stress of 125,000 lbs. wt. or it will fly to pieces.

35. A string, 4 ft. long, in the form of a circle and weighing 2 lbs. revolves about its centre 5 times a second; find its tension.

36. The fly-wheel of an engine revolves 49 times in 11 secs.; if the wheel be 8 ft. in diameter and be made of iron weighing 480 lbs. per cubic foot, find the stress at any point on the rim of the wheel.

37. A particle, of mass 4 lbs., is attached by a light string, $4\sqrt{3}$ ft. long, to the vertex of a right cone, whose vertical angle is 60°. The cone is held with its axis vertical, and the particle revolves as a conical pendulum on the cone, making 140 revolutions in 11 minutes. Find the pressure of the particle on the cone.

38. Assuming the velocity of a point on the equator to be 1520 feet per second, prove that a body loses $\frac{1}{290}$ of its weight approximately by reason of this velocity.

(Radius of earth = 4000 miles.)

CHAPTER XI.

PROJECTILES.

100. IN this chapter we shall consider the motion of bodies projected with any velocity in any direction and acted on by gravity only. A body thus projected is termed a **Projectile**; the angle that the direction of projection makes with the horizontal plane through the point of projection is called **the angle of projection** or **angle of elevation**; the path of the body is termed its **trajectory.**

The distance from the point of projection to the point where the trajectory meets any plane through the point of projection is called the **range** on that plane; and the **time of flight** is the interval between its projection and its striking the plane.

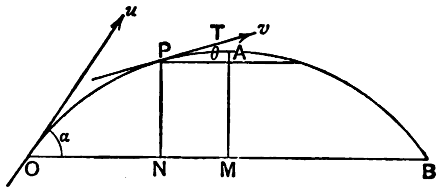
We shall assume throughout that the resistance of the air may be neglected, i.e. that the motion is *in vacuo.* It is well however to state that the resistance of the air to a projectile is very considerable, and alters the character of the trajectory.

We shall also assume that the motion is so near the surface of the earth that the acceleration due to gravity may be considered constant in value.

It is found that a projectile describes a curve, and later on we shall prove that this curve is a parabola.

101. Let the body be projected from the point O with velocity u, and let α be the angle of projection. Let P be the position of the body after t secs., v the velocity then, and

θ the angle its direction of motion at P makes with the horizon.



By the principle of the Physical Independence of Forces the effect of gravity on the motion of the body is entirely in a vertical direction; it has no effect in a horizontal direction. The initial horizontal velocity therefore remains constant, and we have:

Horizontal velocity at time t = u cos α = constant.

Therefore **also horizontal space described in time t**

$$= \mathbf{u \cos \alpha t.} \qquad (s = ut)$$

Again, vertically, the body is subject to an acceleration g *downwards* and it has initially a velocity $u \sin \alpha$ vertically upwards.

Therefore **vertical velocity at time t**

$$= \mathbf{u \sin \alpha - gt,} \qquad (v = u + ft)$$

and **vertical space described in time t**

$$= \mathbf{u \sin \alpha t - \tfrac{1}{2} gt^2.} \qquad (s = ut + \tfrac{1}{2}ft^2)$$

Thus from the figure

$$v \cos \theta = u \cos \alpha,$$

$$v \sin \theta = u \sin \alpha - gt,$$

$$ON = u \cos \alpha t,$$

$$PN = u \sin \alpha t - \tfrac{1}{2} gt^2.$$

Also if s be the vertical space described in time t

$$\mathbf{v^2 \sin^2 \theta = u^2 \sin^2 \alpha - 2gs.} \qquad (v^2 = u^2 + 2fs)$$

102. *To find the velocity and direction of motion at a given time.*

Let v be the velocity required, and θ the angle its direction makes with the horizon.

Then $v \cos\theta =$ horizontal velocity at time t

$$= u\cos\alpha \quad\text{..................(1)}, \qquad \text{(Art. 101)}$$

and $\quad v\sin\theta =$ vertical velocity at time t

$$= u\sin\alpha - gt \quad\text{............(2)}. \qquad \text{(Art. 101)}$$

Squaring (1) and (2), and adding

$$v^2 = u^2\cos^2\alpha + (u\sin\alpha - gt)^2,$$

$$\therefore\ v = \sqrt{u^2 - 2u\sin\alpha \,.\, gt + g^2t^2}.$$

By division, from (1) and (2)

$$\tan\theta = \frac{u\sin\alpha - gt}{u\cos\alpha}.$$

103. *To find the velocity and direction of motion at a given vertical height.*

Let h be the given vertical height, then with the same notation as in the preceding article,

$$v\cos\theta = u\cos\alpha \quad\text{..........................(1)},$$

$$v^2\sin^2\theta = u^2\sin^2\alpha - 2gh,$$

$$\therefore\ v\sin\theta = \sqrt{u^2\sin^2\alpha - 2gh} \quad\text{.................(2)}.$$

By division, from (1) and (2)

$$\tan\theta = \frac{\sqrt{u^2\sin^2\alpha - 2gh}}{u\cos\alpha}.$$

Squaring (1) and (2), and adding

$$v^2 = u^2 - 2gh,$$

$$\therefore\ v = \sqrt{u^2 - 2gh}.$$

104. *To find the greatest height attained.*

Let h be the height required. At the highest point of it's trajectory the body is moving horizontally; hence its vertical velocity is then zero.

$\therefore\ 0 = u^2 \sin^2 \alpha - 2gh,$ $\qquad [v^2 \sin^2 \theta = u^2 \sin^2 \alpha - 2gs.$ Art. 101]

$$\therefore\ h = \frac{u^2 \sin^2 \alpha}{2g}.$$

105. *To find the time to the highest point.*

As in the preceding, at the highest point the vertical velocity is zero; hence if t be the required time,

$0 = u \sin \alpha - gt,$ $\qquad$ [vert. vel. $= u \sin \alpha - gt.$ Art. 101]

$$\therefore\ t = \frac{u \sin \alpha}{g}.$$

COR. The vertical velocity at the *highest* point is zero; therefore the velocity of the body at the highest point $=$ horizontal velocity $= u \cos \alpha$.

106. *To find the range and time of flight on the horizontal plane through the point of projection.*

Let R be the required range, and t the required time of flight. When the body reaches the horizontal plane again, the total vertical space described is zero.

$$\therefore\ 0 = u \sin \alpha t - \tfrac{1}{2} g t^2,$$

whence
$$t = \frac{2u \sin \alpha}{g}.$$

$\therefore\ R =$ horizontal space described in time t

$$= u \cos \alpha t = \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

$$= \frac{u^2 \sin 2\alpha}{g}.$$

COR. 1. At the end of the horizontal range, the vertical velocity of the body

$$= u \sin \alpha - gt$$

$$= u \sin \alpha - 2u \sin \alpha$$

$$= - u \sin \alpha$$

= the initial vertical velocity *reversed in direction.*

COR. 2. *There are two directions of projection which will give the same range;* viz. when the angles of projection are

$$\alpha \text{ and } \frac{\pi}{2} - \alpha.$$

For the range when $\frac{\pi}{2} - \alpha$ is the angle of projection

$$= \frac{u^2}{g} \sin 2\left(\frac{\pi}{2} - \alpha\right) = \frac{u^2}{g} \sin(\pi - 2\alpha)$$

$$= \frac{u^2}{g} \sin 2\alpha$$

= the range when α is the angle of projection.

107. *Given the initial velocity, to find when the horizontal range is a maximum.*

If u be the velocity and α the angle of projection, the range

$$= \frac{u^2 \sin 2\alpha}{g};$$

and is therefore greatest when $\sin 2\alpha$ is greatest,

i.e. when $\sin 2\alpha = 1$,

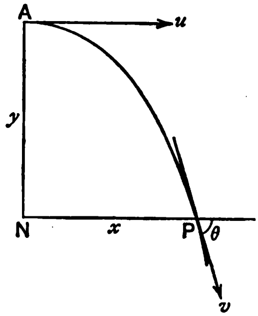
i.e. when $2\alpha = 90°$. $\therefore \alpha = 45°$.

Therefore the maximum range is greatest when the angle of projection is 45°, and the value of the maximum range is

$$\frac{u^2}{g}.$$

108. *The motion of a body projected horizontally.*

Let the body be projected horizontally from the point A with velocity u; let P be its position at time t, y the vertical space, x the horizontal space it has then described; v its velocity at P, and θ the angle its direction of motion then makes with the horizon. By the principle of the Physical Independence of Forces, the effect of gravity is entirely in a vertical direction, and does not affect the horizontal motion.



Therefore for the horizontal motion we have:

$$v\cos\theta = \text{horizontal velocity at time } t = u = \text{constant},$$

$$x = \text{horizontal space described in time } t = ut. \qquad (s = ut)$$

And for the vertical motion:

$$v\sin\theta = \text{vertical velocity at time } t = gt, \qquad (v = u + ft)$$

$$y = \text{vertical space described in time } t = \tfrac{1}{2}gt^2. \quad (s = ut + \tfrac{1}{2}ft)$$

Also $$v^2\sin^2\theta = 2gy. \qquad (v^2 = u^2 + 2fs)$$

109. Ex. i. *A body is projected at an elevation $\sin^{-1}(\frac{3}{25})$ with a velocity of 200 ft. per sec.; find the greatest height attained, and the distance of the body from the point of projection in 3 seconds.*

Let a be the angle of projection, so that the initial vertical velocity
$$=200\sin a \text{ ft. per sec.}$$

When the body is at its highest point, the vertical velocity is zero,
$$\therefore\ 0 = 200^2\sin^2 a - 2gh,$$
where h is the height required.
$$\therefore\ h = \frac{200^2}{64}\times(\tfrac{3}{25})^2 = 9 \text{ feet.}$$

Again, let x be the horizontal space, and y the vertical space described in 3 secs.; then

$$x = 200\times 3\cos a, \qquad (s = ut)$$
$$y = 200\times 3\sin a - \tfrac{1}{2}g\,.\,9. \qquad (s = ut + \tfrac{1}{2}ft^2)$$

Therefore the distance of the body from the point of projection in 3 secs.

$$=\sqrt{x^2+y^2}=\sqrt{(200\times 3\cos a)^2+(200\times 3\sin a-16\times 9)^2}$$
$$=[(200\times 3)^2-32\times 9\times 200\times 3\sin a+(16\times 9)^2]^{\frac{1}{2}}$$
$$=3\left[(200)^2-\frac{32\times 200\times 3\times 3}{25}+(16\times 3)^2\right]^{\frac{1}{2}}$$
$$=3\,[(200)^2-2304+2304]^{\frac{1}{2}}$$
$$=600 \text{ feet.}$$

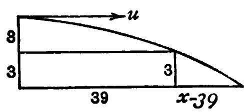
Ex. ii. *A body is projected at an elevation of* 30° *from the top of a cliff with a velocity of* 60 *ft. per sec., and falls into the sea in* 5 *seconds; find the height of point of projection above the level of the sea.*

Let h feet be the height required, then the body describes a vertical height of h ft. *downwards* in 5 seconds,

$$\therefore\ -h=60\sin 30^\circ\times 5-\tfrac{1}{2}g\,.\,25 \qquad (s=ht+\tfrac{1}{2}ft^2)$$
$$=\frac{60\times 5}{2}-16\,.\,25$$
$$=150-400$$
$$=-250 \text{ feet.}$$
$$\therefore\ h=250 \text{ feet.}$$

Ex. iii. *A tennis ball is served horizontally from a height of* 6 *feet, and just clears the net* 3 *feet high and distant* 39 *feet from the server. Find the initial velocity of the ball, and the point where it strikes the ground.*

Let t_1 secs. be the time to the net, t_2 secs. the time to the ground, u the initial velocity of the ball, x the horizontal distance from the server of the point where the ball strikes the ground. When the ball reaches the net the horizontal space described is 39 ft. and the vertical space 3 ft.



$$\therefore\ 39=ut_1\ \ldots\ldots\ldots(1), \qquad (s=ut)$$

and $\qquad 3=\tfrac{1}{2}gt_1^2=16t_1^2.$

Squaring (1), $\qquad 39^2=u^2t_1^2,$

therefore multiplying across $\qquad 3u^2t_1^2=16\times 39^2t_1^2.$

$$\therefore\ u^2=\frac{16\times 39^2}{3},$$
$$u=\frac{4\times 39}{\sqrt{3}}=52\sqrt{3}$$
$$=90 \text{ ft. per sec. approx.}$$

Again the horizontal space described in time $t_2 = x$ ft.,

and the vertical .... $= 6$ ft.

$$\therefore\ x = ut_2 \text{ ....(2)},$$

and $$6 = \tfrac{1}{2}gt_2^2 = 16t_2^2,$$

therefore squaring (2) and multiplying across

$$16x^2 = 6u^2,$$

$$\therefore\ x = 39\sqrt{2} = 55{\cdot}15 \text{ ft. approx.}$$

EXAMPLES. XIII *a*.

1. A body is projected at an elevation of 30° with a velocity of 200 ft. per sec.: find its velocity after 2 secs. and the greatest height attained.

2. A body is projected at an elevation of 75° with a velocity of 100 ft. per sec.: find the horizontal range.

3. A body is projected at an elevation of 30°, and in 3 secs. passes horizontally over a wall: find its initial velocity.

4. A body projected with a velocity of 640 ft. per sec. has a horizontal range of 6400 feet: find the angle of projection.

5. A man throws a cricket-ball at an elevation of 45° with a velocity of 96 ft. per sec.: how far does he throw it?

6. A cannon-ball is fired horizontally from the top of a tower 256 ft. high with a velocity of 300 ft. per sec.: at what distance from the tower will the ball strike the ground?

7. The horizontal range of a projectile is equal to four times the greatest height attained: find the angle of projection.

8. If the greatest height attained by a projectile be equal to three-quarters of the height due to the velocity of projection, find the angle of projection.

9. If the greatest horizontal range of a projectile be 6272 feet, find the velocity of projection.

10. If a be the greatest height, and b the horizontal range of a projectile, find the angle of projection.

11. A particle is projected such that its initial vertical and horizontal velocities are respectively $3g$ and g ft. per sec.: shew as accurately

as possible in a diagram the positions of the particle at the end of each of the first 6 seconds of its motion, and sketch its path during that time.

12. A particle is projected from a given point O in a given direction OA with a velocity of g ft. per sec.; sketch its path by means of a diagram showing the positions of the particle at the end of each of the first 4 seconds of its motion.

13. A body is projected with velocity u and elevation α; prove that it will be moving in a direction at right angles to the direction of projection in $\dfrac{u}{g \sin \alpha}$ seconds.

14. A body is projected with a velocity of 75 ft. per sec. at an elevation $\cos^{-1}(\frac{1}{5})$; find the direction of its motion when its velocity is 30 ft. per second.

15. Prove that the velocity of a projectile projected with a given velocity is least when it reaches its highest point.

16. Prove that a body projected with velocity u, and elevation α, is moving in a direction making an angle β with the horizon after $\dfrac{u \sin (\alpha - \beta)}{g \cos \beta}$ seconds.

17. If t be the time taken by a projectile to reach a point at a horizontal distance x and vertical distance y from the point of projection, show that $gt^2 = 2(x - y)$ if it be projected at an angle of 45°.

18. A man can throw a ball 49 yards vertically upwards: find the greatest horizontal distance he can throw it.

19. A bullet is discharged vertically upwards with a velocity of 352 ft. per sec. from a train which is travelling horizontally at the rate of 60 miles an hour. Neglecting the resistance of the atmosphere, find after what times the bullet will be moving (1) horizontally, (2) in a direction inclined at an angle of 45° to the horizon. Also find the vertical height of the bullet at each of the times above the horizontal plane from which it is fired.

20. A particle is projected at an elevation of 30° with a velocity of 128 ft. per sec.: find (1) the direction of its motion after 2 secs., (2) the greatest height to which it rises.

21. A particle is projected with a velocity whose horizontal and vertical components are 60 and 40 ft. per sec. respectively: find the horizontal range.

22. A bullet is fired with a velocity of 800 ft. per sec.: at what elevation must it be fired in order that it may strike a point in the same horizontal plane at a distance of 10,000 ft.?

23. A body is projected horizontally from the top of a tower $8g$ ft. high with a velocity $4g$ ft. per sec.: find how far from the foot of the tower it strikes the ground, and the direction of its motion at that instant.

24. If the horizontal range of a projectile is equal to the height due to the velocity of projection, find the angle of projection.

25. A body is dropped from a balloon moving horizontally with a velocity of 35 ft. per sec. and reaches the ground in $3\frac{3}{4}$ secs.: find the height of the balloon, and the velocity of the body on striking the ground.

26. A projectile just clears a vertical wall h ft. high and distant h ft. from the point of projection: determine the angle of projection, and the velocity of projection.

27. A body is projected with a velocity of 80 ft. per sec. at an elevation of 45°; with what velocity must a second stone be projected vertically upwards so that both stones may rise to the same height?

28. A ball is projected with a velocity whose horizontal and vertical components are 40 and 94 ft. per sec. respectively: find the magnitude of its velocity after two seconds.

29. The greatest distance to which a boy can throw a cricket-ball is $96\frac{2}{3}$ yards: how long is the ball in the air?

30. A body is projected with a velocity of 32 ft. per sec. at an elevation of 30° from the top of a tower 320 ft. high. Find where it strikes the ground.

31. The time of flight of a projectile on a horizontal plane is 8 secs. and the range is a maximum: find the velocity of projection.

32. A bullet is fired horizontally from a rifle with a velocity of 1200 ft. per sec. and strikes the ground at a distance of 1200 yards: find the height of the rifle above the ground.

33. Shew that for any horizontal range, not the maximum, there are two directions, equally inclined to the direction for maximum projection, at which a particle may be projected with a given initial velocity.

34. The initial horizontal velocity of a projectile is u, its horizontal range is a: shew that the times which elapse between its leaving its initial position and its being at a vertical height h above it are the roots of the equation $ugt^2 - agt + 2uh = 0$.

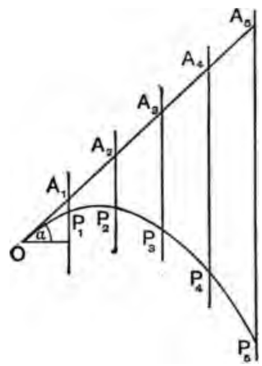
35. A rifle when elevated to the angle whose sine is $\frac{1}{[illegible]}$ has a range of 900 yards on a horizontal plane; shew that, neglecting the resistance of the air, its muzzle velocity is approximately 7[illegible] ft. per sec.

36. A stone is thrown with a velocity 32 ft. per sec. from the top of a tower 32 ft. high at an angle of elevation of 30°. Find where and with what velocity it will strike the horizontal plane through the foot of the tower.

37. How should a stone be thrown from the summit of a vertical cliff 200 ft. high so as to strike the ground at a distance of 200 ft. from the foot of the cliff after an interval of 5 seconds?

38. Find the two directions in which a stone can be thrown with a velocity of 40 ft. per sec. so as to hit a mark 25 ft. distant in the horizontal plane through the point of projection.

110. *A particle is projected with a velocity u, at an elevation α; to represent its path graphically.*



Draw $OA_1A_2\ldots$ in the direction of projection, making

$$OA_1 = A_1A_2 = A_2A_3\ldots = u.$$

The particle *if unaccelerated* would arrive at A_1 in 1 sec., at A_2 in 2 secs., at A_3 in 3 secs. and so on.

But its vertical acceleration, g, brings it down through a vertical space $\frac{1}{2}gt^2$ in time t.

Hence draw vertical lines A_1P_1, $A_2P_2\ldots$ through $A_1A_2\ldots$, and make

$A_1P_1 = \frac{1}{2}g$. $\quad(\frac{1}{2}gt^2$ when $t = 1)$.

$A_2P_2 = \frac{1}{2}g \times 4$. $\quad(\frac{1}{2}gt^2$ „ „ $2)$.

$A_3P_3 = \frac{1}{2}g \times 9$. $\quad(\frac{1}{2}gt^2$ „ „ $3)$.

and so on.

P_1, P_2, P_3, P_4 ... are the positions of the particle after 1, 2, 3, 4 ... secs.

Joining these by an even curve, we have a representation of the path of the projectile.

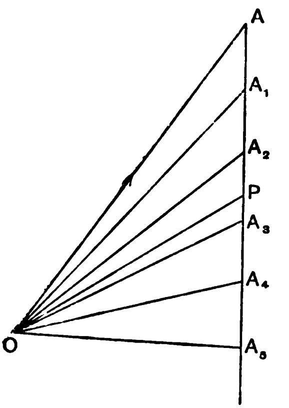
111. *A particle is projected in a given direction with a given velocity: to draw a diagram from which to determine the magnitude and direction of its velocity at any time.*

Draw OA to represent the initial velocity of the particle in magnitude and direction, and from A draw AP vertically downwards to represent gt, the velocity of the particle at time t due to gravity.

Then by the Triangle of Velocities, OP represents the velocity of the particle at time t in magnitude and direction.

Thus if $AA_1 = g$, $AA_2 = 2g$, $AA_3 = 3g$ and so on,

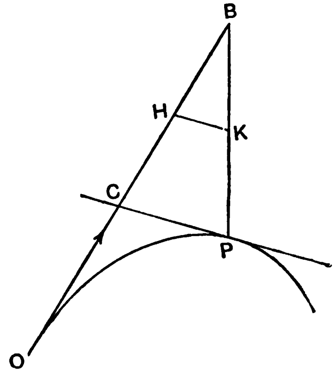
OA_1 represents the velocity of the particle after one second,
OA_2 „ „ „ „ two seconds,
and so on.



112. *Having plotted the path of a particle projected in a given manner, to determine the magnitude and direction of its velocity at any point.*

Let OP be the path, and OB the direction of projection.

It is required to find the magnitude and direction of the velocity of the particle when it is at P.



Draw PB vertically to meet OB at B, and bisect OB at C. Join CP.

From BO cut off $BH = u$ the given initial velocity, and draw HK parallel to CP to meet PB at K.

HK will represent in magnitude and direction the velocity of the particle at P.

Let t be the time to the point P, so that $OB = ut$, $BP = \frac{1}{2}gt^2$.

Then $$\frac{HB}{BK} = \frac{CB}{BP} = \frac{\frac{ut}{2}}{\frac{1}{2}gt^2} = \frac{u}{gt}.$$

But $HB = u$,

$\therefore$ $BK = gt =$ the velocity due to gravity in time t.

$\therefore$ by the Triangle of velocities,

HK represents their resultant velocity in magnitude and direction, the velocity of the particle at P.

Or, the particle when at P is moving along CP with a velocity equal to HK.

EXAMPLES. XIII *b*.

1. Two particles are projected horizontally towards one another with velocities of g and $2g$ ft. per sec. from two points $9g$ feet apart. Draw a figure to represent their paths and hence determine when and where they meet.

2. Two particles are projected from the same point, at the same instant with different velocities, u, v, in the same direction. Draw curves representing their paths and hence shew that the line joining their positions at any instant is parallel to their direction of projection.

3. B is a point distant 40 ft. horizontally and 30 ft. vertically from A, and above it. Two particles are projected simultaneously from A and B towards one another with equal velocities of 10 ft. per sec. Prove that the particles meet and find when and where they do so.

4. A man in a railway carriage travelling 30 miles an hour throws a ball vertically upwards with a velocity of 16 ft. per sec. How far does the train travel before the ball returns to his hand? Draw an accurate diagram to represent the path of the ball in space.

5. A particle is projected horizontally with a velocity of g ft. per sec. Draw a figure to represent its path during the first 5 seconds of its motion.

6. A balloon is travelling horizontally with a velocity of g ft. per sec. From it a man throws a ball with a vertical velocity of $2g$ ft. per sec. Neglecting the atmospheric resistance, draw the path of the ball in space for the first 5 secs. after it leaves the balloon.

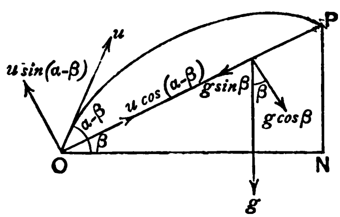
7. A body is projected horizontally with a velocity of g ft. per sec. From a diagram determine the directions of its motion after 1, 2, 3, 4 seconds respectively.

CHAPTER XIV.

PROJECTILES (*continued*).

113. *From a point on a plane inclined at an angle β to the horizon, a particle is projected with velocity u, at an angle α with the horizontal, in a vertical plane through the line of greatest slope: to find* (i) *the time of flight,* (ii) *the range on the plane,* (iii) *the greatest distance of the body from the plane.*

Let t be the time of flight, and R the range.



Velocity u is equivalent to $u\cos(\alpha-\beta)$ along the plane, and $u\sin(\alpha-\beta)$ at right angles to the plane.

The acceleration g of gravity is equivalent to $g\sin\beta$ along the plane and $g\cos\beta$ at right angles to the plane.

Also the motion along the plane may be considered independently of the motion at right angles to the plane.

(i) *Time of flight.*

1st method. When the body reaches the point P on the plane, the space described at right angles to the plane is zero.

$$\therefore\ 0 = u \sin(\alpha - \beta)\, t - \tfrac{1}{2} g \cos \beta t^2. \qquad (s = ut + \tfrac{1}{2} ft^2)$$

$$\therefore\ t = \frac{2u \sin(\alpha - \beta)}{g \cos \beta}.$$

2nd method.

PN = vertical distance described in time t,

ON = horizontal distance described in time t.

$$\therefore\ \tan \beta = \frac{PN}{ON} = \frac{u \sin \alpha t - \tfrac{1}{2} g t^2}{u \cos \alpha t} = \frac{2u \sin \alpha - gt}{2u \cos \alpha},$$

whence
$$t = \frac{2u \sin(\alpha - \beta)}{g \cos \beta}$$
as before.

(ii) *Range.*

$$R = OP = ON \sec \beta = u \cos \alpha \ \sec \beta \,.\, t$$
$$= \frac{2u^2 \cos \alpha \sin(\alpha - \beta)}{g \cos^2 \beta}.$$

(iii) *The greatest distance from the plane.*

Let Y be the distance required.

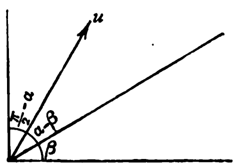
When the body is at its greatest distance from the plane, its velocity at right angles to the plane is zero.

$$\therefore\ 0 = u^2 \sin^2(\alpha - \beta) - 2g \cos \beta \,.\, Y. \qquad (v^2 = u^2 + 2fs)$$

$$\therefore\ Y = \frac{u^2 \sin^2(\alpha - \beta)}{2g \cos \beta}.$$

114 Maximum Range. *To find the direction of projection for the maximum range on an inclined plane.*

By the preceding article, the range



$$= \frac{2u^2 \cos \alpha . \sin(\alpha - \beta)}{g \cos^2 \beta}$$
$$= \frac{u^2 [\sin(2\alpha - \beta) - \sin \beta]}{g \cos^2 \beta}.$$

Hence, u and β being given, we see that the range is greatest when

$$\sin(2\alpha-\beta) \text{ is greatest,}$$

i.e. when
$$\sin(2\alpha-\beta)=1,$$

i.e. when
$$2\alpha-\beta=\frac{\pi}{2},$$

in which case
$$\alpha-\beta=\frac{\pi}{2}-\alpha.$$

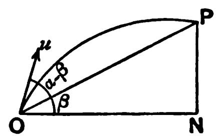
Thus we see from the figure that *the direction of projection for the maximum range bisects the angle between the plane and the vertical.*

The length of the maximum range

$$=\frac{u^2(1-\sin\beta)}{g\cos^2\beta}=\frac{u^2}{g(1+\sin\beta)}.$$

115. *To shew that there are generally* **two directions** *in which a body can be projected from a given point with a given velocity so as to pass through another given point.*

Let O be the point of projection, u the given velocity, α the angle of projection, P the point the body is to pass through.



Through O draw ON horizontally and let angle $PON=\beta$.

Then as in Article 113,

$$OP=\frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}$$

$$=\frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta].$$

Hence
$$\sin(2\alpha-\beta)=\frac{OP\,.\,g\cos^2\beta}{u^2}+\sin\beta,$$

an equation for $2\alpha-\beta$ and therefore for α.

Also, since $\sin\theta = \sin(\pi - \theta)$, two values of $2\alpha - \beta$, each less than π, can be found to satisfy this equation, and if α_1 and α_2 be the two values of α thus found

$$2\alpha_2 - \beta = \pi - (2\alpha_1 - \beta),$$

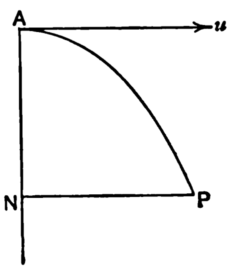
$$\therefore\ \alpha_2 = \frac{\pi}{2} + \beta - \alpha_1.$$

Also since α_1 must obviously be greater than β, α_2 will be an acute angle, and we therefore have two directions in which the body may be projected so as to pass through P.

It will be found that these two directions make equal angles with the line bisecting the angle between OP and the vertical.

116. *A body is projected horizontally from a given point: to prove that it describes a parabola.*

Let the body be projected horizontally from the point A, with velocity u, and let P be its position at time t.



Draw AN vertically downwards and PN horizontally, to meet at N.

The vertical space described in time $t = AN = \frac{1}{2}gt^2$. (Art. 108.)

The horizontal space described in time $t = PN = ut$. (Art. 108.)

$$\therefore\ PN^2 = u^2t^2 = \frac{2u^2}{g}AN.$$

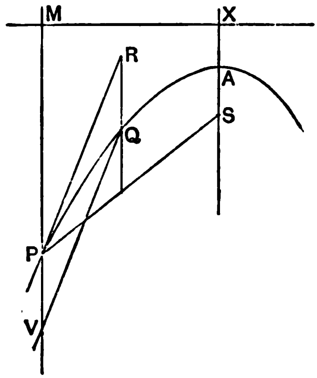
Comparing this with the property of the parabola

$$PN^2 = 4AS\,.\,AN,$$

we see that the body describes a parabola whose vertex is A, whose axis is vertical, and whose latus-rectum $= \dfrac{2u^2}{g}$.

117. *To prove that the path of a body projected in any direction is a parabola.*

Let P be the position at any time of a projectile travelling with velocity u and angular elevation α.



It would, if unaccelerated, travel in time t to the point R, where $PR = ut$.

But its vertical acceleration, g, brings it down through a distance RQ which $= \frac{1}{2}gt^2$.

$\therefore$ Q is its position after a time t.

Complete the parallelogram $PRQV$.

Now $$QV = PR = ut,$$
and $$PV = RQ = \tfrac{1}{2}gt^2.$$

$$\therefore\ QV^2 = u^2t^2 = \frac{2u^2}{g} PV = 4\left(\frac{u^2}{2g}\right) PV.$$

Comparing this with the property of the parabola
$$QV^2 = 4SP \cdot PV,$$
we see that the path of the particle is a parabola, whose axis is vertical, and whose focus S is at a distance from P equal to $\frac{u^2}{2g}$.

118. *The velocity at any point of the parabolic path of a projectile is that due to a fall from the directrix to that point.*

In the figure of the preceding article, draw a vertical PM to meet the directrix in M.

Suppose a particle to fall from M to P; then its velocity at P

$$= \sqrt{2g.PM} = \sqrt{2g.SP} = \sqrt{2g.\frac{u^2}{2g}} = u$$

$$= \text{the velocity of the projectile at } P.$$

119. *To find the latus-rectum of the parabolic path of a projectile.*

Using the notation and figure of the two previous articles draw SAX vertically to meet the path in A and the directrix in X.

Then the velocity at the vertex A of the parabola

$$= \text{that due to a fall through height } XA.$$

$$\begin{aligned} \therefore \sqrt{2g.AX} &= \text{velocity at vertex (the highest point)} \\ &= \text{constant horizontal velocity} \\ &= u \cos \alpha. \end{aligned}$$

$$\therefore \text{latus-rectum} = 4AX = \frac{2u^2 \cos^2 \alpha}{g}.$$

120. *To find the focus of the parabolic path of a projectile.*

With the figure of Art. 117, on the opposite side of PR make the angle RPS equal to the angle MPR, and cut off SP equal to $\frac{u^2}{2g}$.

S shall be the focus.

For since the tangent to a parabola bisects the angle between the focal radius and the perpendicular on the directrix, the focus lies in PS.

Also the focal distance of $P = \frac{u^2}{2g}$ (Art. 117).

$$\therefore S \text{ is the focus required.}$$

121. *To find the equation of the path of a projectile referred to horizontal and vertical axes through the point of projection.*

Taking P as the point of projection, and using the notation and figure of Art. 117, let (x, y) be the co-ordinates of Q.

Then $x =$ horizontal distance described in time $t = u \cos \alpha t$, $y =$ vertical distance described in time $t = u \sin \alpha t - \frac{1}{2}gt^2$.

Hence eliminating t by substitution from the first equation

$$y = x \tan \alpha - \frac{gx^2 \sec^2 \alpha}{2u^2},$$

the required equation.

Cor. The above equation may be written

$$gx^2 \sec^2 \alpha - 2xu^2 \tan \alpha = -2u^2 y,$$

i.e. $$x^2 - x\frac{u^2}{g} \sin 2\alpha = -\frac{2u^2 y \cos^2 \alpha}{g},$$

i.e. $$\left(x - \frac{u^2 \sin 2\alpha}{2g}\right)^2 = -\frac{2u^2 \cos^2 \alpha}{g}\left[y - \frac{u^2 \sin^2 \alpha}{2g}\right].$$

From this we see that the latus-rectum of the parabola is $\frac{2u^2 \cos^2 \alpha}{g}$ and that its vertex is at the point

$$\left(\frac{u^2 \sin 2\alpha}{2g}, \frac{u^2 \sin^2 \alpha}{2g}\right).$$

122. *A particle, starting with velocity u in a given direction, is subject to a constant acceleration f, in a constant direction other than that of its initial velocity: to prove that it describes a parabola.*

The proof of Art. 117 holds for this throughout, if we write 'f' instead of 'g' the acceleration of gravity; and remember that 'vertical' now becomes 'the direction of the constant acceleration f.'

As an example of motion of this kind, we may take the case of a body projected *along an inclined plane* of elevation β, at an angle γ with the line of greatest slope.

The component of the acceleration of gravity at right angles to the plane ($g \cos \beta$) is balanced by the pressure on the plane, and the body moves on the plane subject to an acceleration $g \sin \beta$ parallel to the line of greatest slope.

Thus, if u be the initial velocity,

the horizontal velocity at time t $= u \sin \gamma$,

the horizontal space described in time $t = u \sin \gamma t$,

the velocity at time t parallel to the line of greatest slope

$$= u \cos \gamma - g \sin \beta t,$$

the space described in time t parallel to the line of greatest slope

$$= u \cos \gamma t - \tfrac{1}{2} g \sin \beta t^2.$$

123. Ex. i. *A ball projected with velocity u strikes a vertical wall at right angles. If the wall be at a horizontal distance k from the point of projection, find the direction of projection of the ball.*

Let the ball be projected at elevation a, and let t be the time to the wall.

When the ball strikes the wall the motion is horizontal, therefore the vertical velocity is then zero, hence

$$0 = u \sin a - gt. \qquad (v = u + ft)$$

$$\therefore\ t = \frac{u \sin a}{g}.$$

Also $k =$ the horizontal distance described in time t

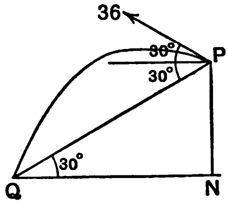
$$= u \cos at = \frac{u^2 \sin a \cos a}{g},$$

$$\therefore\ \sin 2a = \frac{2gk}{u^2},$$

$$\therefore\ a = \tfrac{1}{2} \sin^{-1}\left(\frac{2gk}{u^2}\right).$$

Ex. ii. *A body is projected from a point on an inclined plane of elevation* 30° *with a velocity of* 36 *ft. per second, at an angle of* 30° *with the horizontal: find the range down the plane.*

Let t be the time of flight, and PQ the range as shewn in the figure. Draw QN horizontally, PN vertically.



The acceleration at right angles to the plane $= -g \cos 30°$.

Also when the body arrives at Q, the space described at right angles to the plane is zero.

$\therefore\ 0 = 36 \sin 60°\, t - \frac{1}{2} g \cos 30°\, t^2$, $(s = ut + \frac{1}{2} f t^2)$

whence $\quad t = \frac{9}{4}$ seconds.

Also the range

$= PQ = QN \sec 30°$

$= \sec 30° \times$ (horizontal dist. described in time t)

$= \sec. 30° \times 36 \cos 30°\, t$

$= 36\, t$

$= 81$ feet.

124. Second Method (See Art. 115) of proving that:

There are generally two directions in which a body can be projected from a given point with a given velocity so as to pass through another given point.

The equation of the path in Art. 121 may be written

$$2u^2 y = 2u^2 x \tan \alpha - g x^2 (1 + \tan^2 \alpha)$$

i.e. $$g x^2 \tan^2 \alpha - 2u^2 x \tan \alpha + g x^2 + 2u^2 y = 0.$$

This is a quadratic for $\tan \alpha$, therefore if the point (x, y) is given we generally obtain two values of $\tan \alpha$.

125. *Third method* (**Geometrical**) *of proving the same proposition.*

Let the particle projected from P with velocity u be required to pass through the point Q.

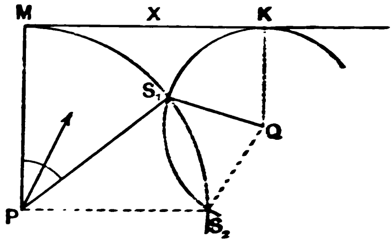
Draw PM vertically upwards and equal to $\frac{u^2}{2g}$. Also draw MX horizontally. MX is the directrix of the path of the particle.

Draw QK perpendicular to MX.

With centres P and Q and radii PM, QK describe circles.

Two circles generally cut at two points. Let these cut at S_1 and S_2.

Then since $S_1P = PM$, and $S_1Q = QK$, a parabola described with focus S_1 and directrix MK passes through P and Q.



Also the tangent to this parabola at P bisects $\angle S_1PM$.

$\therefore$ if a particle is projected along this tangent with the given velocity it will pass through Q.

In the same way, if the particle is projected with the given velocity along the line bisecting $\angle S_2PM$, it will pass through Q, which proves the proposition.

126. *A particle is projected from a given point P with a velocity represented in magnitude and direction by PQ. To find, geometrically, the focus, directrix and latus rectum of its parabolic path, the greatest height to which the particle rises, and its horizontal range on the plane through the point of projection.*

If h is the height of the directrix above the point P,

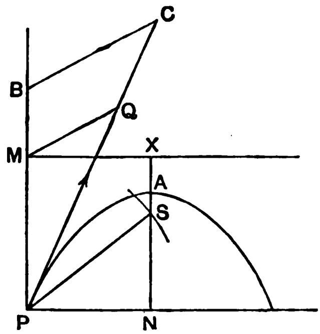
$$u^2 = 2gh,$$

i.e. $$PQ^2 = 2gh,$$

and to determine h we have to find a third proportional to $2g$, and PQ.

Draw PB vertically upwards, making $PB = PQ$.

Along PQ take $PC = 2g$. Join BC, and draw QM parallel to CB meeting PB at M.



From the similar Δs PMQ, PBC, $\dfrac{PM}{PB} = \dfrac{PQ}{PC}$.

$$\therefore\ PM = \frac{PQ^2}{PC} = \frac{u^2}{2g}, \qquad \text{i.e. } PM = h.$$

$\therefore$ the str. line MX drawn horizontally through M, is the directrix.

Again, PQ, the tangent at P to the path, bisects the angle between PM the perpendicular on the directrix and the focal distance of P.

$\therefore$ on the opposite side of PQ to the point M, making the angle $QPS = \angle QPM$, and cutting off $SP = PM$, S is the focus.

Through S draw SX perpendicular to the directrix and bisect it at A.

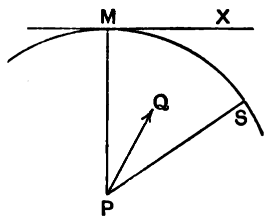
Also draw PN horizontally to meet XS produced at N.

A being the vertex of the parabola, AN is the greatest height to which the particle rises, and its range on the horizontal plane through $P = 2PN$.

Lastly, $2SX$ = the latus-rectum of the path.

127. *A number of particles are projected from the same point P in different directions but with equal velocities. To prove that their parabolic paths have a common directrix.*

Let the particles be projected from the point P with velocity u.



Draw PM vertically upwards equal to $\frac{u^2}{2g}$, and draw MX horizontally.

Then M is a point on the directrix of the path of each particle, for the magnitude and direction of PM are independent of the direction of projection of each particle.

$\therefore$ MX is a common directrix to all the paths.

A number of particles are projected from the same point P in different directions but with equal velocities. To prove that the locus of the foci of their parabolic paths is a circle whose centre is P.

With the same construction as above, describe a circle MS with centre P and radius PM.

Let PQ be the direction of projection of any one of the particles, and on the opposite side of PQ to M, make

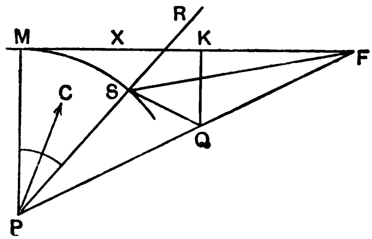
$$\angle QPS = \angle QPM.$$

Then since $SP = PM$ and $\angle SPQ = \angle MPQ$, S is the focus of the path of this particle.

$\therefore$ the circle MS is the locus of the foci of the paths of all the particles, for PM is constant.

128. *A particle is projected from a point P with a given velocity in a given direction, to find geometrically its range on an inclined plane PQ through the point of projection.*

Let the particle be projected in direction PC.



As in Art. 126 find MX the directrix and S the focus of its path.

Let PQ meet the directrix in F. Join SF.

Producing PS to R, make $\angle FSQ = \angle FSR$.

Draw QK perpendicular to the directrix.

Since SF bisects the ext. $\angle RSQ$, $\dfrac{SQ}{SP} = \dfrac{FQ}{FP}$

$$= \frac{QK}{PM} \text{ (from similar } \triangle\text{s } FQK,\ FPM).$$

But $SP = PM$, $\therefore\ SQ = QK$.

$\therefore$ Q is a point on the path of the particle.

$\therefore$ PQ is the range required.

N.B. The construction employed here may be found in any text-book on Geometrical Conics.

If the particle in the previous article is projected **in any direction,** we see that its range PQ on the inclined plane is a maximum when PSQ is a straight line, i.e. when S lies in PQ.

In this case the direction of projection bisects the angle MPQ, as proved by another method in Art. 114.

EXAMPLES. XIV.

1. A plane is inclined to the horizon at an angle of 30°; from a point on it a body is projected at an elevation of 60° with the horizon, and with a velocity of 48 ft. per second: find the time of flight and the range on the plane.

2. From a point on a plane inclined at 30° to the horizon a particle is projected at right angles to the plane with velocity u: find its range on the plane.

3. A particle is projected horizontally with velocity u from a point on a plane inclined at 45° to the horizon: find its time of flight, and range on the plane.

4. A particle is projected with velocity u, and elevation α, from a point on a plane inclined at angle β to the horizon: find when it is moving parallel to the plane, and its distance from the plane at that instant.

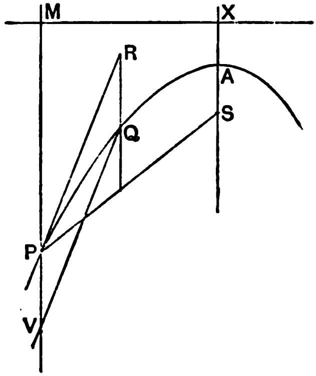
5. A particle is projected from a point on a plane inclined to the horizon at an angle of 30° with a velocity of 160 ft. per sec.: find its greatest range on the plane.

6. Find the range and time of flight of a particle projected with a velocity of 80 ft. per sec. from a point on a plane inclined at an angle $\tan^{-1}\frac{3}{4}$ to the horizon; the angle of elevation of the projectile being 45°.

7. A particle is projected in vacuo under the action of gravity at an elevation α up an inclined plane of elevation β, and strikes the plane at right angles. Prove that

$$\tan\alpha \sin\beta \cos\beta = 1 + \sin^2\beta.$$

Let P be the position at any time of a projectile travelling with velocity u and angular elevation α.



It would, if unaccelerated, travel in time t to the point R, where $PR = ut$.

But its vertical acceleration, g, brings it down through a distance RQ which $= \frac{1}{2}gt^2$.

$\therefore$ Q is its position after a time t.

Complete the parallelogram $PRQV$.

Now $$QV = PR = ut,$$

and $$PV = RQ = \tfrac{1}{2}gt^2.$$

$$\therefore\ QV^2 = u^2t^2 = \frac{2u^2}{g}PV = 4\left(\frac{u^2}{2g}\right)PV.$$

Comparing this with the property of the parabola

$$QV^2 = 4SP.PV,$$

we see that the path of the particle is a parabola, whose axis is vertical, and whose focus S is at a distance from P equal to $\frac{u^2}{2g}$.

118. *The velocity at any point of the parabolic path of a projectile is that due to a fall from the directrix to that point.*

In the figure of the preceding article, draw a vertical PM to meet the directrix in M.

Suppose a particle to fall from M to P; then its velocity at P

$$=\sqrt{2g\,.\,PM}=\sqrt{2g\,.\,SP}=\sqrt{2g\,.\,\frac{u^2}{2g}}=u$$

$$= \text{the velocity of the projectile at } P.$$

119. *To find the latus-rectum of the parabolic path of a projectile.*

Using the notation and figure of the two previous articles draw SAX vertically to meet the path in A and the directrix in X.

Then the velocity at the vertex A of the parabola

$$= \text{that due to a fall through height } XA.$$

$$\begin{aligned} \therefore\ \sqrt{2g\,.\,AX} &= \text{velocity at vertex (the highest point)} \\ &= \text{constant horizontal velocity} \\ &= u\cos\alpha. \end{aligned}$$

$$\therefore\ \text{latus-rectum} = 4AX = \frac{2u^2\cos^2\alpha}{g}.$$

120. *To find the focus of the parabolic path of a projectile.*

With the figure of Art. 117, on the opposite side of PR make the angle RPS equal to the angle MPR, and cut off SP equal to $\dfrac{u^2}{2g}$.

S shall be the focus.

For since the tangent to a parabola bisects the angle between the focal radius and the perpendicular on the directrix, the focus lies in PS.

Also the focal distance of $P=\dfrac{u^2}{2g}$ (Art. 117).

$$\therefore\ S \text{ is the focus required.}$$

26. The angular elevation of a fort on a hill h ft. high is β: shew that in order to hit it the initial velocity of a projectile must not be less than

$$\sqrt{gh\,(1+\operatorname{cosec}\beta)}.$$

27. Prove that the locus of the foci of the paths of all projectiles passing through two given points is an hyperbola.

28. A particle is projected with a velocity of 40 ft. per sec. and when 10 ft. above the horizontal plane through the point of projection its vertical velocity is 10 ft. per sec. Shew that the latus rectum of its path is about 53·75 feet.

29. A number of particles are projected in a vertical plane from the same point with the same velocity but with different angles of projection: find the locus of the foci of their paths.

30. Two bodies are projected in the same vertical plane simultaneously from the same point, with velocities v_1, v_2, and angles of elevation a_1, a_2. Shew that their distance apart after a time t is

$$t\sqrt{v_1^2+v_2^2-2v_1v_2\cos(a_1-a_2)}.$$

31. In a parabolic orbit find the equation of the focal chord through the point of projection, referred to horizontal and vertical axes through that point. Find also the time of flight between the two extremities of this chord.

32. Two heavy particles are projected in the same vertical plane at the same instant with equal velocities from two fixed points, and meet. Shew that the sum of the inclinations to the horizon of the directions of projection of the two particles must be constant.

33. Each of two projectiles is moving directly towards the other at a certain instant; shew that they must ultimately meet.

34. Two projectiles are describing the same parabola, in the same direction; prove that their horizontal distance asunder remains always the same.

35. A projectile is shot with a velocity of a mile per minute at an elevation of 30°: find its greatest height.

How much less would the greatest height attained be, if the angle of elevation were 10″ less?

36. Give a construction for the direction in which a body may be projected with a given velocity from a given point that its path may pass through another given point. Shew that the problem is not always possible.

*37. A coach wheel, rolling with uniform velocity, throws off continually small particles of dust at a tangent to its circumference: find the position on the wheel from which the particles will rise to the greatest height.

[*The following problems are to be solved geometrically.*]

38. PQ is a line of greatest slope on an inclined plane. A particle projected from P impinges on the plane at Q at right angles; find the direction of projection and the directrix of the path of the particle.

39. A particle projected from P in the given direction PR passes through Q. Bisect PQ at V, and draw VR vertically to meet PR at R. Prove that RQ is the direction of the particle's motion when at Q.

40. A particle is projected from a given point O with a given velocity represented by OA. Give a simple geometrical construction to determine the *time* when the particle is moving in a given direction OC.

41. PQ is a given inclined plane, and PQ is the maximum range of a particle projected from P with a given velocity. Find geometrically the direction of the particle's motion when it is at Q.

42. AP is a chord of a vertical circle, A being its highest point. A particle slides down AP from rest at A, and leaves the chord at P. Prove that the horizontal line through A is the directrix of its parabolic path, and that the focus lies in the radius of the circle through P.

43. A particle is projected from a given point P in a given direction, and its path touches the given line QR. Find geometrically the point where it touches QR. Find also the directrix of the path.

CHAPTER XV.

REVISION PAPER. XV *a*.

1. Two trains, of lengths 108 feet and 90 feet, moving in opposite directions on parallel rails were observed to be 3 seconds in completely passing one another, the velocity of the longer being double that of the other. Find the velocity of each train in miles per hour.

2. A ball starts with a uniform velocity of 3 ft. per second, and at the end of each second has a new velocity impressed upon it, at right angles to its direction of motion and equal in magnitude to its velocity during that second. Draw accurately the path of the ball, using squared paper, and determine its distance from the starting point after 4 seconds.

3. A man raises a bucket whose mass is m lbs. from the bottom of a well. When the bucket has risen h ft. its velocity is v feet per sec. How much work has the man done up to this time?

4. A point P is moving along a plane curve with velocity v; at a certain instant the point and the tangent to the curve through it are at distances r, p respectively from a fixed point O. What is then its angular velocity about O?

5. A body whose mass is 4 lbs. is fastened to a string 8 ft. long, and is whirled round so that the string describes a horizontal circle; if the string would break under a tension of 10 lbs. wt., what is the greatest speed in integral ft. per sec. which can be given to the body?

6. A particle is projected with a velocity of 128 ft. per sec. at an angle at 45° with the horizon. Find graphically the magnitude and direction of its velocity after 1 sec., after 2 secs., and after 3 secs. Verify one result by calculation.

REVISION PAPER. XV *b*.

1. Two candles stand on a table, one of which is lighted and burns down at the rate of 2 inches an hour. Find the rate at which the shadow of the other rises on a vertical wall, the plane of the candles being perpendicular to the wall, and the distance of the unlighted candle from the wall being nine-tenths of the distance of the lighted candle.

2. Particles are let fall from the same spot at intervals of half-a-second. How far has one fallen when the next starts, and how far apart are these two when the first has fallen 32 feet?

3. A train of 150 tons wt., moving with a velocity of 50 miles an hour, has its steam shut off and its brakes applied, and is stopped in 363 yds. Supposing the resistance to its motion to be uniform, find its value, and find also the mechanical work done by it measured in foot-pounds, to the nearest thousand.

4. Supposing the resistance of the air on a falling raindrop to be proportional to the square of its speed, show that for a certain speed the acceleration is zero, so that a drop that has attained this speed will continue to fall with constant speed.

If the resistance of the air lessens the downward acceleration of the drop by 2 ft.-sec. units when the speed is 10 ft. per sec., what is the constant speed?

5. The moon has a mass of $1 \cdot 7 \times 10^{24}$ pounds, and describes a circular orbit about the earth of radius $1 \cdot 26 \times 10^{9}$ feet in 27 days 8 hours. Calculate the moon's acceleration and the force that produces it. State your units.

6. A particle in vacuo has an initial velocity v, and is projected from a given point P. Give a geometrical construction for finding the direction of projection when the particle passes through a given point Q.

REVISION PAPER. XV *c*.

1. A and B are two points 15 feet apart. A ball moves from B, at right angles to AB, with a uniform velocity of 4 ft. per sec., and starting at the same instant a second ball travels from A with a uniform velocity of 5 ft. per sec. If the balls collide, find the direction in which the second ball travels.

2. The mass of a body is 15 lbs. and its velocity is 20 ft. per second; find its kinetic energy in foot-poundals. Find also the number of lbs. wt. in the force which would bring it to rest in one-tenth of a second.

3. Equal weights each of mass $2m$ are attached to the ends of a fine string which passes over two fixed smooth pulleys and under a smooth moveable pulley of mass m; the pulleys being so arranged that all the parts of the string which are not in contact with the pulleys are vertical. Find the upward acceleration of the moveable pulley, and the tension of the string.

4. A point is observed to have a velocity of 25 ft. per sec. in a certain direction, and three seconds afterwards to have a velocity of 21 ft. per sec. making an angle of 48° with its original direction. Find

the magnitude and direction of the additional velocity impressed upon it during the three seconds, and assuming it is subject to a uniform acceleration, find the value of that acceleration.

5. A *watt* is 10^7 ergs per second. Show that a kilowatt is approximately equal to $1\frac{1}{3}$ horse-power. [1 lb. = 460 grams, 1 foot = 30 cms., $g=32$.]

6. A light string 13 ft. long is fastened at one end to a fixed point 5 ft. above a smooth horizontal table, and at the other to a mass of 4 lbs. The mass describes a circle on the table at a uniform speed of 6 ft. per sec. Find the pressure on the table.

REVISION PAPER. XV *d*.

1. A train is travelling at the rate of 15 miles an hour, when a man jumps out of it at right angles to its direction with a velocity of 4 ft. per second. In what direction does he fall?

2. A particle moves from a given point in a straight line with a uniform velocity of 4 ft. per sec., and one second after it has started a second particle starts from rest at the same point in the same straight line with an acceleration of 2 ft.-sec. units. Find graphically when the second particle overtakes the first. Give your result to the nearest tenth of a second.

3. Water at 750 lbs. per sq. in. pressure acts on a piston one sq. ft. in area, through a stroke of one foot; what is the work that such water does per cubic foot? and per gallon? If an hydraulic company charges 18 pence for a thousand gallons of such water, how much work is given for each penny? [1 gal. = 10 lbs. wt., 1 c. ft. = 1000 oz. wt.]

4. A man in a moving lift finds that a weight of 10 lbs. only indicates 9 lbs. when weighed by means of a spring balance. Investigate the motion of the lift.

5. A stone is projected from the top of a tower with velocity v, and at an angle of elevation a; at the same instant, another stone is dropped from the same place. Find the relative velocity and acceleration of the first stone with respect to the second.

6. A particle is projected from a point A with a given velocity in a given direction. If B is any point in its path, C the middle point of AB, and D the point where the vertical through C cuts the path, prove that $CD = \frac{g}{2}t^2$, where t is the time from A to D.

REVISION PAPER. XV *e*.

1. Two ships collide, their directions of motion immediately before collision making an angle of 105° with one another. If the velocities of the ships are equal, find the direction of the blow.

2. At A, B masses of 4 and 5 lbs. are respectively placed, and each mass is acted upon by a force equal to the weight of the other, and in a direction from it towards the other. If they start at the same moment, and meet at the end of $2\frac{1}{2}$ seconds, find the distance AB, and the velocities of the masses at the moment of striking.

3. An hydraulic company charges 18 pence per thousand gallons of water at 750 lbs. pressure per sq. inch ; how much is this per horse-power hour? [1 gal. = 10 lbs. wt., 1 c. ft. = 1000 oz. wt.]

4. A body hangs from the roof of a railway carriage by equal strings attached to two points one directly in front of the other so that the strings are at right angles. If the acceleration of the train is 4 ft.-sec. units, find the ratio of the tensions of the strings.

5. Find the greatest velocity (in miles per hour) at which an engine of 192 horse-power can pull a train of 150 tons mass along the level, the frictional and other resistances to motion being 16 lbs. wt. per ton.

6. A shot is fired from a gun at the top of a cliff of height h with a velocity of u ft. per sec. Prove that if the range measured from the foot of the cliff is as great as possible, the elevation a is given by the equation

$$\cos 2a = \frac{gh}{u^2+gh}.$$

REVISION PAPER. XV *f*.

1. A man in a railway carriage drops a ball from a height of 4 feet above the floor. If the train is travelling at the rate of 30 miles an hour, find the direction of the ball in space at the instant of striking the floor.

2. A weight of W tons is moved from rest by a force of P tons, till its velocity is v; it is then brought to rest by a force of Q tons, and altogether has passed over a ft. in t seconds.

Prove (1) that $$t^2 = \frac{2aW}{g}\left(\frac{1}{P}+\frac{1}{Q}\right)$$

(2) that its kinetic energy, when its velocity is v, is

$$PQa \div (P+Q).$$

3. A body which weighs 10 lbs. is attached to a cord passing over a fixed smooth pulley, and the cord is drawn over the pulley at a uniform rate of 3 ft. per sec. What is the magnitude of the tension of the string?

If the pulley is rigidly attached to a moving platform, which is ascending with an acceleration of 4 ft.-sec. units, and the cord is drawn over the pulley as before, what change is made in the tension?

4. The position of a point P on a straight line is given by $x=2t^2-3t+5$, where t denotes the number of seconds after a fixed instant of time, and x the distance in feet of P from a fixed point O on the line. Find the velocity of the point at the end of t seconds, and its acceleration.

5. A steamer at A which cannot sail more than 8 knots, wishes to intercept a steamer at B doing 11 knots in a direction making an angle of 40° with BA. Find how much choice of direction she has without failing to intercept, and give to the nearest degree, the angle within which her course must lie.

6. A mass m is placed at a distance r from the centre of a rough horizontal revolving disc. If μ is the coefficient of friction, find the greatest angular velocity the disc can have if the mass is not to change its position on the disc.

REVISION PAPER. XV*g*.

1. A particle moves in a straight line under a retardation of 6 ft.-sec. units, starting with a velocity of 24 ft. per second. Draw a graph of its velocity-time equation, and from your diagram determine the space described in 4 secs. and in 6 secs.

2. A mass of 12 lbs. is attached to a string which is held in the hand; if the hand be lowered with a uniform acceleration of 10 ft. per second per second, what is the force tending to stretch the string?

3. A turbine is driven by a stream flowing at the rate of 5 miles an hour, the size of the stream being such that 300 gallons of water enter the turbine every second. Find the horse-power of the turbine, its efficiency being taken to be 60 per cent. [1 gal. = 10 lbs. wt.]

4. A particle moves in a straight line with a uniform acceleration of 0·5 ft.-sec. units, the initial velocity being 0·7 ft. per sec. Construct on squared paper a diagram, in which the ordinate represents the space passed over in a given time, 2 inches on the paper representing a foot, and also representing a second.

Show that the tangent of the inclination to the time-axis of a line through the points on the diagram corresponding to $\frac{9}{10}$ and $1\frac{1}{10}$ seconds is equal numerically to the velocity at the end of one second. Give reasons for this equality.

5. When one of the keys of a pianoforte is depressed through $\frac{3}{8}$ of an inch, the hammer on the pianoforte is raised through a vertical height of 2 inches. A force of 2 ozs. wt. will just depress the key. Find the weight of the hammer. (Neglect all weights except that of the hammer.)

If the key is depressed through the distance mentioned with a uniform force of 6 ounces wt., find the velocity imparted to the hammer.

6. Two particles are simultaneously projected from the same point A with equal velocities in directions AT, AT', where AT' makes the lesser angle with the vertical, and the parabolas intersect at B. Determine which particle arrives first at B.

Prove also that the product of their times of transit from A to B is equal to the square of the time occupied by a particle falling from rest vertically through a space equal to AB.

REVISION PAPER. XV *h*.

1. A rope unwinds from a drum, and the hanging end descends vertically at the rate of 10 ft. per sec.; with what velocity does the centre of gravity of the hanging part of the rope descend?

2. A train runs from rest at one station and stops at the next. During the first quarter of the journey the motion is uniformly accelerated, and during the last quarter it is uniformly retarded, the middle portion being performed at a uniform full speed. Prove that the average speed of the train is $\frac{2}{3}$ of the full speed.

3. Six equal weights (W lbs.) are fastened to a rope in such a way that one follows another at distances of one foot. The rope hangs vertically with the lowest weight 3 ft. above the ground; if the rope is gradually lowered, draw a diagram for the work done on the bodies, when all have come to the ground.

4. A mass of 1000 lbs. is moving in a straight line at a speed of 30 miles an hour. Its energy is reduced to 500 ft.-lbs. in 12 seconds. Calculate the retardation and the force which acted to produce it.

5. A particle of 24 lbs. is fastened to the vertex of a cone by a string 8 ft. long. If the semi-vertical angle of the cone is $\cos^{-1}\frac{5}{13}$ and the particles revolve on the surface of the cone (whose axis is vertical) at a uniform speed of 8 ft. per second, find the pressure on the cone.

6. A particle is projected at an elevation a and strikes an inclined plane through the point of projection at right angles: prove that the inclination θ of the plane to the horizontal is given by the equation

$$2\tan(a-\theta)=\cot\theta.$$

CHAPTER XVI.

IMPULSE AND COLLISION.

129. Def. *An impulsive force is a very great force which acts for a very short time, and its effect is measured by the change of momentum produced.*

Thus if an impulsive force, acting upon a body of mass m, change its velocity from u to v, the effect of the impulsive force, or its impulse (I) as it is usually called, is measured by $mv - mu$,

i.e. $$I = mv - mu.$$

The Second Law of Motion justifies the above definition, for it tells us that the total effect of a force is measured by the change of momentum produced.

Since the time during which an impulsive force acts is *very* short, the change in position of the body in the time during which the force acts may be neglected.

It must be remembered that impulsive forces are of precisely the same nature as ordinary forces, but they have to be treated differently on account of our inability to measure very small intervals of time and to observe what takes place during those intervals.

As illustrations of impulsive forces we may take any sudden blow, as the blow of a hammer; two balls colliding with one another; or a bullet striking a target.

130. If any force P act for an interval of time t on a mass m, and change its velocity from u to v in that time, its

total effect is measured (according to the Second Law of Motion) by the change of momentum produced in that time,

i.e. by
$$\begin{aligned} & mv - mu, \\ &= m(v-u) \\ &= mft, \text{ where } P = mf. \qquad (v = u + ft) \\ &= Pt. \end{aligned}$$

This holds good for all values of P and t, and hence if P be a *very large* force acting for a *very short* time t its total effect or its impulse I will still be

$$mv - mu = Pt.$$

Example. A 1 oz. bullet strikes a wooden target with velocity of 1600 ft. per second, and is brought to rest in $\frac{1}{10}$ second.

The measure of the impulse

$$\begin{aligned} &= I = \text{momentum destroyed} \\ &= \tfrac{1}{16} \cdot 1600 \\ &= 100 \text{ units of impulse.} \end{aligned}$$

Also if P be the average force exerted by the bullet on the target whilst it is being brought to rest,

$$I = P \cdot \tfrac{1}{10} \; (I = Pt), \quad \text{i.e.} \quad P = 1000 \text{ poundals.}$$

It is well to notice the fact that the product of two quantities, one infinitely large and the other infinitely small, may be finite.

Thus the product of 10000 and $\frac{1}{10000}$ is unity, i.e. measure of the impulse of 10000 poundals acting for $\frac{1}{10000}$th part of a second is unity. $(I = Pt)$

In the same way if 10^{10} poundals act for $\dfrac{1}{10^{10}}$ of a second its total effect is measured by unity.

131. Motion of shot and gun.

In this case the explosion of the powder produces the impulsive force. By the Third Law of Motion the action on the shot is equal and opposite to the reaction on the gun,

i.e. the impulse on the shot is equal and opposite to that on the gun.

Hence the momentum of the shot forwards

= the momentum of the gun backwards.

132. Ex. i. *A body of mass* 10 *lbs. has its velocity suddenly changed from* 12 *to* 16 *ft. per sec. Find* (1) *the measure of the impulse,* (2) *the measure of the force if the blow lasted for* $\frac{1}{100}$*th of a second.*

Let I be the impulse, and P the force required. Then

$$I = \text{change of momentum produced}$$
$$= 10\,(16 - 12)$$
$$= 40 \text{ units of impulse.}$$

Also since $I = Pt$,

$$P = 40 \div \tfrac{1}{100} = 4000 \text{ poundals.}$$

Ex. ii. *A shot of* 15 *lbs. mass leaves a gun of* 1500 *lbs. mass with a muzzle velocity of* 1000 *ft. per sec.; how far will the gun recoil up an inclined plane rising* 1 *in* 8?

Let v be the initial velocity of recoil of the gun, a the angular elevation of the plane, s the space required.

The momentum of the gun equals momentum of shot, therefore

$$1500v = 15 \times 1000,$$
$$v = 10 \text{ ft. per sec.}$$

Also the acceleration of the gun on the plane

$$= -g \sin a = -\tfrac{32}{8} = -4 \text{ ft.-sec. units,}$$

$$\therefore\ 0 = 10^2 - 2 \,.\, 4 \,.\, s, \qquad (v^2 = u^2 + 2fs)$$

i.e. $$s = 12\tfrac{1}{2} \text{ ft.}$$

EXAMPLES. XVI*a*.

1. A mass of 5 lbs. at rest is struck and moves off with a velocity of 20 ft. per sec.: find the impulse, and the mean value of the force if the blow lasted for $\frac{1}{100}$ sec.

2. An impulse, whose measure is 360, changes the velocity of a body from 12 ft. per sec. to 36 ft. per sec.: find the mass of the body.

3. A body of mass 12 lbs. moving with a velocity of 15 ft. per sec. has its velocity reversed in direction but unaltered in magnitude by a blow: find the measure of the impulse.

4. A ball, of mass m, strikes a wall and rebounds with a velocity equal to one-half of its striking velocity (u); find the impulse, and the average value of the force if the blow lasted for $\frac{1}{n}$th of a second.

5. A marble of mass 2 ozs., let fall upon a pavement from a height of 9 feet, rebounds to a height of 4 feet; find the measure of the impulse.

6. A mass starts with a velocity of 20 ft. per sec.: with what velocity will a mass four times as great start under the action of an impulse twice as great?

7. Two bodies move off with velocities of 30 and 25 ft. per sec. under the action of two impulses whose measures are as 45 is to 50: compare the masses of the bodies.

8. A ball of mass m falling from rest under gravity strikes the ground after a certain time t, and on its rebound rises to half its original height. Find the measure of the impulse of the ground.

9. Two masses of 3 lbs. and 4 lbs. are fired from the same gun with equal charges of powder: compare the velocities with which they leave the gun.

10. A mass projected vertically upwards under a given impulse rises to a height of 64 ft. and a second mass under an impulse 4 times as great takes 3 secs. to reach its highest point: compare the two masses.

11. A shot of mass 1 lb. is projected from a gun of mass 140 lbs. with a muzzle velocity of 2000 ft. per sec.: find the initial velocity of recoil of the gun.

✓ 12. A shot of 800 lbs. is fired from a 40-ton gun with a velocity of 2100 ft. per sec.: find the steady pressure which will destroy the velocity of recoil of the gun in 5 feet.

13. A gun, of mass half-a-ton, fires a shot of 14 lbs. with a velocity of 1200 ft. per sec.: how far will the velocity of recoil carry the gun up a plane rising 1 in 4?

14. A steam hammer weighing 15 tons falls from a height of 4 feet upon a mass of red-hot iron, and is brought to rest in $\frac{1}{16}$th of a second after touching the iron. Find the impulse, the mean force exerted by the hammer, and how much it flattens the iron.

15. A gun of 60 tons projects a bolt of 4 cwt. The gun recoils 2 ft. up an inclined plane of 30°; find the initial velocity of the shot.

16. A shot weighing 500 lbs. is fired with a velocity of 1600 ft. per sec. from a gun weighing 35 tons. Find the velocity with which the gun recoils. If the recoil be resisted by a steady pressure equal to a weight of 10 tons, to what distance will the gun move?

133. Elasticity. When a body is compressed it has a tendency to resume its original shape as soon as the compressing force is removed, and it is found that all bodies possess this property in a greater or less degree; i.e. all bodies are more or less **elastic.**

The internal force which a body thus brings into play in order to resume its original shape is called the **Force of Restitution.**

When two spheres collide or impinge, the line joining their centres at the instant of impact is called the *line of impact.* If the centres of the spheres move in the line of impact, the impact is said to be *direct;* if not, the impact is said to be *oblique.*

134. Newton's Experimental Law of Elasticity.

Newton discovered, by experiment, that when two bodies impinge **their relative velocity in the line of impact after the collision bears a constant ratio to their relative velocity in the same direction before the collision, but is opposite in direction, i.e. opposite in sign.**

This constant ratio is known as '*the Coefficient of Elasticity,*' '*the Modulus of Elasticity,*' '*the Coefficient of Restitution or Resilience,*' and is generally denoted by e. It varies with the substances of which the bodies are made, but is independent of their masses. Thus if two steel balls impinge, e will be constant whatever the size of the balls; but it will have a different value when two cork balls are used, or when one ball is ivory and the other steel.

When the coefficient of restitution is zero the body is said to be *inelastic,* and when the coefficient is equal to unity the body is said to be *perfectly elastic.*

In nature no bodies are perfectly elastic or perfectly

inelastic, and in all cases the value of e is found to lie between 1 and 0.

We shall in this chapter consider the impact of spheres and particles against one another and against planes only. In all cases we shall suppose the bodies to be smooth.

135. Direct impact on a fixed plane.

A smooth ball or particle whose coefficient of elasticity is e, impinges directly on a fixed plane with velocity u; to find its velocity on leaving the plane.

The effect of the impact is entirely in the line of impact, therefore the ball after striking the plane will return along its line of impact.

Let it return with velocity v.

Since the plane is fixed, the relative velocity of the ball to the plane *after* impact $= v$, and the relative velocity of the ball to the plane *before* impact $= u$, and these are in opposite directions.

Therefore by Newton's Experimental Law of Elasticity,

$$v = eu,$$

i.e. the ball after impact returns along its line of impact with velocity eu.

COR. 1. If the body be inelastic, $e=0$ and therefore $v=0$, i.e. the body will not leave the plane after striking it.

If the body be perfectly elastic, $e=1$ and therefore $v=u$, i.e. the velocity of the body is reversed but unchanged in value by the impact.

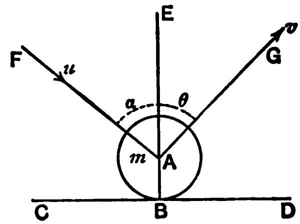
COR. 2. If m be the mass of the particle, when the body strikes the plane its momentum mu is destroyed, and momentum mv is generated in the opposite direction.

Therefore the measure of the impulse of the ball = total change of momentum

$$= mu + mv = mu\,(1 + e).$$

136. Oblique impact on a fixed plane.

Let a ball A impinge on a fixed smooth plane CBD striking it at B.



Let the ball move with velocity u along FA, making angle α with the normal to the plane, before impact, and with velocity v along AG, making angle θ with the normal, after impact.

The effect of the impact is entirely along the line of impact, i.e. at right angles to the plane.

Therefore the motion parallel to the plane is unaltered by the impact.

Hence $$v \sin\theta = u \sin\alpha \quad \text{.....................(i).}$$

Also by Newton's Experimental Law of Elasticity,

$$v \cos\theta = eu \cos\alpha \quad \text{..................(ii).}$$

[The velocity of the ball relative to the plane before impact is $u \cos\alpha$, and the velocity of the ball relative to the plane after impact is $v \cos\theta$ in the opposite direction.]

By division, from (i) and (ii)

$$\tan\theta = \frac{\tan\alpha}{e},$$

or $$\cot\theta = e \cot\alpha.$$

Also squaring and adding (i) and (ii)

$$v^2 = u^2 (\sin^2\alpha + e^2 \cos^2\alpha)$$

$$v = u \sqrt{\sin^2\alpha + e^2 \cos^2\alpha}.$$

We have thus found the velocity, in magnitude and direction, of the ball after impact.

Cor. 1. If the ball be inelastic $e = 0$,

$$\therefore\ v \cos\theta = 0,$$

i.e. $\cos\theta = 0$ and $\theta = \frac{\pi}{2}$, which shews that the body will not leave the plane after impact.

From equation (i) we see that it will move along the plane with velocity $u \sin\alpha$.

COR. 2. If the ball be perfectly elastic $e=1$, and we have from equations (i) and (ii)

$$\tan\theta = \tan\alpha,$$
$$\theta = \alpha \text{ and } v = u.$$

Hence if a perfectly elastic body impinge on a fixed smooth plane its velocity is unaltered in magnitude by the impact, and the angle of reflection is equal to the angle of incidence.

COR. 3. The impulse on the plane is measured by the change of momentum on the ball, the change of momentum being measured in the line of impact.

Now momentum $mu\cos\alpha$ before impact is destroyed, and momentum $mv\cos\theta = meu\cos\alpha$ after impact is generated in the opposite direction.

Therefore I (the impulse) = total change of momentum

$$= mu\cos\alpha + meu\cos\alpha$$
$$= mu\cos\alpha\,(1+e).$$

137. When two masses impinge, the action on one is equal and opposite to the action on the other, by the Third Law of Motion, therefore the impulse on one is equal and opposite to the impulse on the other, i.e. the change of momentum (measured in the line of impact) of the one is equal and opposite to the change of momentum (also measured in the line of impact) in the other. Hence the sum of these two changes, measured in the same direction, is zero.

Thus the total change of momentum produced by the impact is zero.

Therefore when two smooth masses impinge the total momentum (measured in the line of impact)

after impact is equal to their total momentum (measured in the same direction) before impact.

Again, the impulse produces no effect in a direction at right angles to the line of impulse, and there is therefore no change of momentum in that direction. It follows therefore that the momentum after impact *in any direction* is equal to that before impact in the same direction.

138. The preceding may be very easily proved analytically thus. Let u_1, u_2 be the velocities of the bodies in the line of impact before the collision, v_1, v_2 their velocities in the same direction after collision, m_1, m_2 their masses; I the measure of the impulse between them.

Considering the mass m_1,

$$I = \text{change of momentum produced}$$
$$= m_1(u_1 - v_1).$$

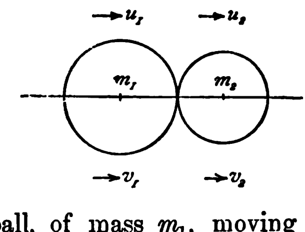
Considering the other mass m_2

$$-I = \text{change of momentum produced}$$
$$= m_2(u_2 - v_2),$$
$$\therefore\ m_1u_1 - m_1v_1 = -m_2u_2 + m_2v_2,$$

i.e.
$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2,$$

or *Total momentum after impact = total momentum before impact.*

139. Direct impact of two smooth balls.



A smooth ball, of mass m_1, moving with velocity u_1, impinges directly on another smooth ball, of mass m_2, moving in the same direction with velocity u_2. To find the velocities

of the balls after impact, the coefficient of restitution being e.

Let v_1, v_2 be the velocities of the balls after impact.

By Newton's Law of Elasticity, the relative velocity after impact $= (-e)$ times that before impact.

$$\therefore\ v_1 - v_2 = -e(u_1 - u_2)\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

By the Third Law of Motion, there is no momentum lost in the line of impact (Art. 100).

$$\therefore\ m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Multiply (i) by m_2 and adding

$$(m_1 + m_2)\,v_1 = (m_1 - em_2)\,u_1 + m_2\,(e+1)\,u_2,$$

$$\therefore\ v_1 = \frac{(m_1 - em_2)\,u_1 + m_2\,(e+1)\,u_2}{m_1 + m_2}.$$

Also multiplying (i) by m_1 and subtracting

$$(m_1 + m_2)\,v_2 = m_1\,(e+1)\,u_1 + (m_2 - em_1)\,u_2,$$

$$\therefore\ v_2 = \frac{m_1\,(e+1)\,u_1 + (m_2 - em_1)\,u_2}{m_1 + m_2}.$$

We have thus found v_1 and v_2 the velocities of the balls after impact.

N.B. If the balls be moving in *opposite* directions before impact, their relative velocity will then be $u_1 + u_2$, and the momentum before impact will be $m_1u_1 - m_2u_2$.

COR. i. If $I =$ the impulse on the first ball,

$$I = m_1(u_1 - v_1)$$

$$= \frac{m_1m_2}{m_1 + m_2}(u_1 - u_2)(1 + e).$$

COR. ii. If the balls be inelastic $e = 0$, and equation (i) becomes

$$v_1 - v_2 = 0.$$

This shows us that when two inelastic balls impinge directly, they move on together after impact.

It will be seen later on (Arts. 149, 150) that **when an impulse takes place, some Energy is dissipated or lost,** except when the elasticity is perfect. Such dissipated energy is expended in the generation of heat and sound. The point for the student to remember is that in Impulse Problems the Principle of Conservation of Energy is inapplicable, except when the elasticity is perfect.

The Principle of Conservation of Momentum can be applied universally.

EXAMPLES. XVI *b*.

1. Two balls impinge directly and move on together after impact. What do you deduce?

2. A traction engine moving with a certain velocity runs into a wall and knocks it down. Why does not a man have the same effect on the wall if he runs into it with the same velocity?

3. A mass of 5 oz. impinges on a smooth fixed plane at right angles with a velocity of 26 ft. per sec., and rebounds with a velocity of 10 ft. per sec. What is the measure of the impulse?

4. A ball of elasticity e is dropped from a height h above a fixed horizontal plane. To what height will it rebound from the plane? Deduce the height to which it will rebound after striking the plane a second time.

5. A ball A impinges obliquely upon a ball B at rest. In which direction will the ball B move off?

6. Prove that if a perfectly elastic ball impinges obliquely upon a smooth fixed plane, its angle of reflection is equal to its angle of incidence, and its velocity is unchanged in magnitude.

7. An inelastic ball impinges upon a smooth fixed plane with a velocity u and at an angle α with the normal to the plane. Find its velocity after impact.

8. A perfectly elastic ball impinges at right angles to a fixed smooth plane with a velocity u. Find the measure of the impulse.

9. A perfectly elastic ball impinges obliquely upon a fixed smooth plane with a velocity u, at an angle α with the normal to the plane. Find the measure of the impulse.

10. Two balls moving at right angles impinge so that the line of impact is in the direction of the motion of one of the balls. What do you see as to the direction of the motion of this ball after impact?

11. AB, BC are lines of greatest slope of two inclined planes facing one another, ABC being an obtuse angle. An inelastic ball slides down AB, and strikes BC with velocity u. Find its velocity after impact.

12. How could a ball be made to travel along the sides of a regular hexagon with uniform velocity?

13. A projectile strikes a vertical plane. What difference does the impact make in the time of flight of the ball from its starting point to the ground again?

14. If a man were placed upon a perfectly smooth horizontal table, and could not reach an edge with either hand or foot, how could he get off the table?

15. A ball of mass 5 lbs. moving with a velocity of 8 ft. per sec., impinges directly on a ball of mass 4 lbs. moving in the same direction with a velocity of 4 ft. per sec.: find the velocities of the balls after impact when $e=\frac{1}{4}$.

16. A ball of mass $4m$, moving with a velocity of 8 ft. per sec., impinges directly on a ball of mass m moving in the opposite direction with a velocity of 4 ft. per sec.: find the velocities of the balls after impact when $e=\frac{1}{3}$.

17. Two balls, of masses $2m$ and m, moving with velocities of 5 ft. per sec. and 4 ft. per sec. respectively in opposite directions, impinge directly, and the heavier ball is reduced to rest: find the coefficient of restitution.

18. A ball impinges directly upon an equal ball at rest; shew that the velocities of the balls after impact are as $1-e : 1+e$, where e is the coefficient of elasticity.

19. Two balls, whose masses are as 2 to 1, whose respective velocities before impact are as 1 to 2 in opposite directions, and whose elasticity is $\frac{5}{6}$, impinge directly: shew that each ball will move back after impact with $\frac{5}{6}$ths of its original velocity.

20. Three perfectly elastic balls, whose masses are as $1:3:6$, move with equal momenta in a straight line. Shew that each of the two smaller balls will be reduced to rest when it overtakes the ball in front of it.

21. Three inelastic balls, of masses m, $2m$, $3m$ respectively are placed (not in contact) in a straight line. The first impinges on the second with velocity $6u$; find the velocity acquired by the third when it is struck.

22. Two balls impinge directly, and the impact interchanges their velocities: prove that they are perfectly elastic, and of equal mass.

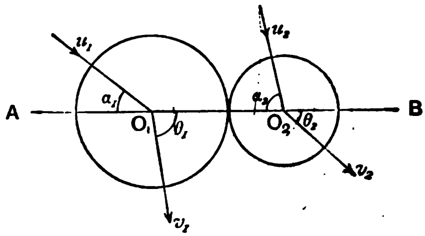
23. Two perfectly elastic balls, masses m and m', moving in the same direction, strike each other. If the hindmost ball be reduced to rest by the blow, shew that its velocity must have been more than double that of the other.

24. A number of perfectly elastic equal balls is placed in a straight line; the first is made to impinge on the second with velocity u, the second strikes the third, and so on: find the velocity of the nth ball.

25. A number of perfectly elastic balls, of masses m, $2m$, $4m$, $8m$..., is placed in a straight line; the first impinges directly on the second with velocity u, the second strikes the third, and so on; find the velocity of the nth ball.

140. Oblique impact of two smooth balls.

Let m_1 and m_2 be their masses; AB the line of impact through their centres O_1, O_2; e the coefficient of elasticity.



Given u_1, u_2 their velocities before impact, α_1, α_2 the inclinations of their directions to AB before impact: to find v_1, v_2 their velocities after impact, and θ_1, θ_2 the inclinations of their directions to AB after impact.

The effect of the impact is entirely in the line of impact, i.e. the velocities of the balls at right angles to AB are unaltered by the impact,

$$\therefore\ v_1 \sin\theta_1 = u_1 \sin\alpha_1 \ \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)},$$

and

$$v_2 \sin\theta_2 = u_2 \sin\alpha_2 \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

By Newton's Law of Elasticity,

[Relative velocity *in the line of impact* after impact $= (-e)$ times that before.]

$$v_1 \cos\theta_1 - v_2 \cos\theta_2 = -e\,(u_1 \cos\alpha_1 - u_2 \cos\alpha_2) \ldots\ldots \text{(iii)}.$$

By the Third Law of Motion, there is no momentum lost, *in the line of impact*: therefore

$$m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 = m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2 \ldots \text{(iv)}.$$

We thus have four equations from which to determine the four unknowns v_1, v_2, θ_1, θ_2.

Multiplying (iii) by m_2 and adding to (iv) we have,

$$v_1 \cos\theta_1 = \frac{m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2 - e m_2 (u_1 \cos\alpha_1 - u_2 \cos\alpha_2)}{m_1 + m_2}$$

$$\ldots\ldots\ldots \text{(v)}.$$

Similarly

$$v_2 \cos \theta_2 = \frac{m_1 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2 + e m_1 (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)}{m_1 + m_2} \quad \text{.........(vi).}$$

We now obtain v_1 by squaring and adding (i) and (v), v_2 by squaring and adding (ii) and (vi).

Dividing (i) by (v) we obtain $\tan \theta_1$, and dividing (ii) by (vi) we obtain $\tan \theta_2$.

Thus all four quantities v_1, v_2, θ_1, θ_2 are determined.

COR. 1. The impulse on the mass m_1 = its change of momentum in the line of impact

$$= m_1 (u_1 \cos \alpha_1 - v_1 \cos \theta_1)$$

$$= \frac{m_1 m_2}{m_1 + m_2} (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)(1 + e)$$

= the impulse on the mass m_2 in the opposite direction.

COR. 2. If the ball m_2 be at rest before impact $u_2 = 0$,

$$\therefore \sin \theta_2 = 0, \text{ i.e. } \theta_2 = 0 \text{ from (ii)},$$

or the mass m_2 will move off in the line of impact.

We might see this also from first principles; for a ball at rest when struck must move off in the direction of the blow.

141. Action during impact. When two bodies impinge, each becomes compressed to a certain extent, and subsequently if they be elastic they recover to a greater or less degree their original shapes. We may therefore divide the action during impact into two periods, (1) *the period of compression*, (2) *the period of restitution*. The action between the bodies during the former period is called *the force of compression*, that during the latter period *the force of restitution*. At the instant of greatest compression the bodies have a common velocity.

To show that the force of restitution = e times the force of compression, where e is the coefficient of elasticity between two balls.

Let I denote the *force of compression* and I' the *force of restitution*; u_1, u_2 the velocities of the balls before impact; v_1, v_2 their velocities after impact; u their common velocity at the instant of greatest compression; m_1, m_2 the masses of the balls.

Then

$I =$ loss of momentum of m_1 during the period of compression

$= m_1(u_1 - u)$ ....(i)

$=$ gain of momentum of m_2 during the period of compression

$= m_2(u - u_2)$ ....(ii).

From (i) $\qquad u_1 - u = \dfrac{I}{m_1}$,

(ii) $\qquad u - u_2 = \dfrac{I}{m_2}$;

therefore adding (to eliminate u)

$$u_1 - u_2 = I\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \text{ ....(iii).}$$

Again,

$I' =$ loss of momentum of m_1 during the period of restitution

$= m_1(u - v_1)$

$=$ gain of momentum of m_2 during the period of restitution

$= m_2(v_2 - u)$.

Hence, as before,

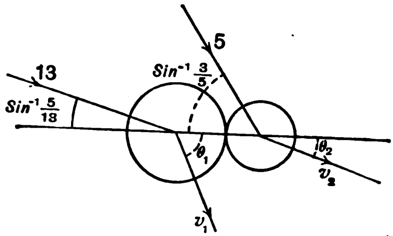
$$v_2 - v_1 = I'\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \text{ ....(iv),}$$

therefore from (iii) and (iv)

$$\frac{I'}{I} = \frac{v_2 - v_1}{u_1 - u_2} = e,$$

i.e. $\qquad I' = e\,.\,I.$

142. *Example. Two balls of mass* 2 *lbs. and* 1 *lb. moving with velocities of* 13 *and* 5 *feet per sec. impinge obliquely. If their coefficient of elasticity is* ·5, *and they make angles* $\sin^{-1}\frac{5}{13}$, *and* $\sin^{-1}\frac{3}{5}$ *with the line of impact before striking, find their velocities and directions after impact.*



Let v_1, v_2 be their velocities after impact making angles θ_1, θ_2 with their line of impact.

$$\sin^{-1}\tfrac{5}{13}=\alpha=\cos^{-1}\tfrac{12}{13}.\quad \sin^{-1}\tfrac{3}{5}=\beta=\cos^{-1}\tfrac{4}{5}.$$

The effect of the impact is entirely in the line of impact,

$$\therefore\ v_1\sin\theta_1=13\sin\alpha=5 \ldots\ldots\ldots\ldots(1),$$

and

$$v_2\sin\theta_2=\ 5\sin\beta=3 \ldots\ldots\ldots\ldots(2).$$

By Newton's Law of Elasticity,

$$\begin{aligned} v_1\cos\theta_1-v_2\cos\theta_2 &= -\tfrac{1}{2}(13\cos\alpha-5\cos\beta)\\ &= -\tfrac{1}{2}(13\times\tfrac{12}{13}-5\times\tfrac{4}{5})\\ &= -4 \ldots\ldots\ldots\ldots(3). \end{aligned}$$

There is no momentum lost in the line of impact,

$$\begin{aligned} \therefore\ 2v_1\cos\theta_1+v_2\cos\theta_2 &= 2\times 13\cos\alpha+5\cos\beta\\ &= 2\times 13\times\tfrac{12}{13}+5\times\tfrac{4}{5}\\ &= 28 \ldots\ldots\ldots\ldots(4). \end{aligned}$$

Adding (3) and (4) $\quad 3v_1\cos\theta_1=24,$

$$v_1\cos\theta_1=8 \ldots\ldots\ldots\ldots(5),$$

$\therefore$ from (1), by division, $\quad \tan\theta_1=\frac{5}{8}=\cdot625$

and $\quad \theta_1=32^\circ$ (from Mathl. Tables).

Squaring (1) and (5) and adding,

$$v_1^2=64+25=89.$$

$$\therefore\ v_1=9\cdot 43 \text{ ft. per sec.}$$

Again, from (3) and (5) $v_2 \cos\theta_2 = 12$ ....(6),

$\therefore$ from (2) and (6) $\tan\theta_2 = \frac{3}{12} = \cdot 25$

and $\theta_2 = 14° \, 2'$.

Squaring (2) and (6) and adding,

$$v_2^2 = 9 + 144 = 153,$$

$$\therefore \; v_2 = 12\cdot 37 \text{ ft. per sec.}$$

EXAMPLES. XVI *c*.

1. A ball, moving with velocity u, impinges obliquely, at an angle of 30° with the line of centres, on a ball of equal mass at rest: find the velocities and directions of the balls after impact, when the balls are perfectly elastic.

2. A ball, of mass $2m$, impinges obliquely at an angle of 45° with the line of centres on a ball of mass m at rest; shew that after impact its direction makes an angle $\tan^{-1} \frac{3}{2}$ with the line of impact, the balls being inelastic.

3. Two perfectly elastic equal balls move in directions at right angles to each other and impinge: shew that their directions of motion after impact are at right angles to one another.

4. Two perfectly elastic balls, of masses $2m$ and m, impinge obliquely with equal velocities, their directions before impact making angles of 60° with the line of centres and with one another: find the velocities and directions of the balls after impact.

5. Two perfectly elastic equal balls impinge obliquely; their velocities are equal, and make angles of 30° and 60° with the line of impact: find the magnitude and direction of their subsequent velocities.

6. $ABCD$ is a square, and two particles P and Q are moving from B to A and from D to A respectively; the mutual action takes place along the line BA, the restitution is perfect, and the masses and velocities are equal: shew that P will be brought to rest and that Q will move along CA produced.

7. One imperfectly elastic ball strikes another at rest; what must be the ratio of their masses in order that the two balls may move in directions at right angles to one another after impact?

8. Two equal imperfectly elastic balls moving with equal velocities (u) in opposite parallel directions impinge on each other, their lines of motion being inclined at an angle α to the line joining their centres at the instant of impact. Determine their subsequent motions, and find the value of α when the path of each is deviated through a right angle.

9. The diameter of each of two equal balls is 2 inches, and the balls are moving in opposite directions, each with velocity u, along two parallel straight lines 1 inch apart. If the coefficient of elasticity be $\frac{3}{4}$, find the velocity and direction of motion of each ball after impact.

10. Find the direction and magnitude of the impulse which will change the direction of motion of a mass m by a given angle α, without altering the magnitude of its velocity, u being the magnitude of that velocity.

11. Two equal particles, having elasticity e, are projected towards one another in the same vertical plane with the same velocity u and at the same elevation α from two points in a horizontal plane at a distance a apart. Prove that each ball makes an angle

$$\tan^{-1}\left(\frac{u\sin\alpha\cos\alpha - ag}{eu\cos^2\alpha}\right)$$

with the horizon immediately after they impinge.

12. Two balls impinge obliquely. If u_1, u_2 are their velocities immediately before impact, making angles α_1, α_2 with their line of impact, prove that their relative velocity immediately after impact makes an angle

$$\tan^{-1}\left(\frac{u_1\sin\alpha_1 - u_2\sin\alpha_2}{e(u_2\cos\alpha_2 - u_1\cos\alpha_1)}\right)$$

with their line of impact.

Jerks of Inelastic Strings.

143. *Example. A mass of* 4 *lbs. after falling vertically through* 16 *feet, jerks a mass of* 6 *lbs. off a table by means of an inelastic string passing over a smooth pulley. Both portions of the string being vertical at the time of the jerk, find the velocity with which the* 6 *lb. mass leaves the table, and the impulsive tension of the string.*

The velocity of the falling ball immediately before the jerk

$$= \sqrt{2g \cdot 16} = 32 \text{ ft. per sec.}$$

The string being inelastic, the balls have a common velocity after the jerk. Let v be this velocity immediately after the jerk. No momentum is lost by the jerk,

$$\therefore (6+4)v = 4 \times 32$$

and $$v = 12{\cdot}8 \text{ ft. per second.}$$

The impulsive tension of the string = the momentum generated in the 6 lb. mass $= 6 \times 12{\cdot}8 = 76{\cdot}8$ units of momentum.

EXAMPLES. XVI *d*.

1. Masses of 6 and 8 lbs. lie on a smooth table side by side and are attached to the ends of a light inelastic string. If the 8 lb. mass is projected along the table with a velocity of 10 ft. per sec., find the velocity of each body after the string tightens.

2. A ball is projected vertically upwards with a velocity of 40 ft. per sec. and, after it has risen 9 feet, lifts an equal mass from rest by means of a light inelastic string. Find the velocity with which this latter mass starts.

3. Masses of 7 and 8 lbs. lie on a smooth table side by side and are connected by a light inelastic string. If the 8 lb. mass is projected along the table with a velocity of 10 ft. per sec., find the velocity of each body after the string is taut.

Also find the impulsive tension of the string.

4. Masses of 6 and 2 lbs. lie close to one another on a table and are connected by a light string 7 ft. long. Another string is fastened to the 6 lb. mass and passing over a smooth pulley is fastened at the other end to a mass of 8 lbs. When this string is just taut, the system is allowed to move. What is the velocity of each mass after the 2 lb. mass is jerked off the table? The strings are supposed to be inelastic.

5. A mass of 8 lbs. is attached to a mass of 4 lbs. by a light inelastic string. The 8 lb. mass is at rest vertically below a small pulley fixed at the top of a smooth inclined plane falling 3 in 10, and the string passes over this pulley. The 4 lb. mass slides down the plane from rest, and when it has been in motion for 5 seconds, the string tightens. Find the velocity of each body immediately after this.

6. A mass of 4 lbs. after falling freely through 4 feet lifts a mass of 12 lbs. from rest vertically upwards by means of a light inelastic string passing over a smooth fixed pulley. How far will the 12 lb. mass rise?

What is the impulsive tension of the string when the jerk takes place?

144. *A ball falls from a height h on a horizontal plane and rebounds, falls again and rebounds, and so on: find the total space described, and the time before the ball ceases to rebound.*

Let e be the coefficient of restitution.

The velocity acquired in falling from height h

$$= \sqrt{2gh}. \qquad (v^2 = u^2 + 2fs)$$

Let $$u = \sqrt{2gh}.$$

Therefore the velocity immediately after the first rebound $= eu$.

Therefore the body strikes the plane a second time with velocity eu, and rebounds a second time with velocity e^2u.

Similarly, it strikes the plane a third time with velocity e^2u, and rebounds a third time with velocity e^3u, and so on.

The time taken to fall to the ground the first time $= \frac{u}{g}$.

$$(v = u + ft)$$

The time taken by gravity to destroy velocity $eu = \frac{eu}{g}$.

$$(v = u + ft)$$

The time taken by gravity to destroy velocity $e^2u = \frac{e^2u}{g}$, and so on.

Therefore the total time required

$$= \frac{u}{g} + \frac{2eu}{g} + \frac{2e^2u}{g} + \frac{2e^3u}{g} + \ldots\ldots \text{ ad inf.}$$

$$= \frac{u}{g}[1 + 2e(1 + e + e^2 + \ldots)] \quad \text{(Geom. Progression)}$$

$$= \frac{u}{g}\left[1 + \frac{2e}{1-e}\right] = \frac{u}{g}\left(\frac{1+e}{1-e}\right)$$

$$= \sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right).$$

Again, the body leaves the ground after the first impact with velocity eu, therefore it rises to height

$$\frac{(eu)^2}{2g} = e^2h. \qquad (v^2 = u^2 + 2fs)$$

Similarly after the second impact it rises to height

$$\frac{(e^2u)^2}{2g} = e^4h,$$

and after the third impact it rises to height

$$\frac{(e^3u)^2}{2g} = e^6h,$$

and so on.

Therefore the total space described

$$= h + 2\,[e^2h + e^4h + e^6h + \dots \text{ ad inf.}]$$
$$= h + 2e^2h\,[1 + e^2 + e^4 + \dots]$$

(Geom. Progression)

$$= h + \frac{2e^2h}{1 - e^2} = h\left(\frac{1 + e^2}{1 - e^2}\right).$$

145. *Assuming that rain falls freely from a height of* 729 *feet, find the pressure per acre due to a fall of* 3 *inches in* 24 *hours.* (*A cubic foot of water weighs* 1000 *ozs.*)

[N.B. The pressure on an acre is equal to the total momentum *per second* destroyed by the reaction of the ground.]

The velocity on striking the ground

$$= \sqrt{2g \times 729} \qquad (v^2 = u^2 + 2fs)$$
$$= 8 \times 27 \text{ ft. per sec.}$$

The volume of rain that falls on an acre in 24 hours

$$= 9 \times 4840 \times \tfrac{1}{4} \text{ cub. ft.},$$

therefore the mass of rain that falls on an acre in 24 hours

$$= \frac{9 \times 4840 \times 1000}{4 \times 16} \text{ lbs.},$$

therefore the mass that falls per sec. on an acre

$$= \frac{9 \times 4840 \times 1000}{4 \times 16 \times 24 \times 60 \times 60} \text{ lbs.},$$

therefore momentum destroyed per sec.

$$= \frac{9 \times 4840 \times 1000 \times 8 \times 27}{4 \times 16 \times 24 \times 60 \times 60} \text{ units of momentum.}$$

Therefore pressure on an acre

$$=\frac{9\times 4840\times 1000\times 8\times 27}{4\times 16\times 24\times 60\times 60}\text{ poundals}$$

$$=\frac{9\times 4840\times 1000\times 8\times 27}{4\times 16\times 24\times 60\times 60\times 32}\text{ lbs. wt.}$$

$$=53\tfrac{89}{512}\text{ lbs. wt.}$$

146. *A continuous jet of water is thrown by a fire-engine so as to strike a wall at right angles with a velocity of* 80 *ft. per sec. If the section of the hose be* 4 *square inches, and the water rebound with a velocity of* 10 *ft. per sec., find the pressure on the wall.*

The mass of water that reaches the wall *per second*

$$=\tfrac{4}{144}\times 80\times \tfrac{1000}{16}\text{ lbs.}$$

Total *change* of velocity due to the impact

$$=80+10=90\text{ ft. per sec.,}$$

therefore momentum destroyed *per second*

$$=\frac{4\times 80\times 1000\times 90}{144\times 16}\text{ units of momentum.}$$

Therefore pressure on the wall

$$=\frac{4\times 80\times 1000\times 90}{144\times 16}\text{ poundals}$$

$$=\frac{4\times 80\times 1000\times 90}{144\times 16\times 32}\text{ lbs. wt.}$$

$$=390\tfrac{5}{8}\text{ lbs. wt.}$$

147. *A pile of mass* 12 *cwt. is driven vertically into the ground by a mass of* 6 *cwt. falling on it from rest through a space of* 9 *feet. If the mean resistance of the ground to penetration by the pile be equal to* $2\frac{1}{10}$ *tons wt., find the distance through which the pile is driven at each blow, the pile being considered inelastic.*

The velocity with which the falling mass strikes the pile

$$= \sqrt{2g \,.\, 9} = 24 \text{ ft. per sec.} \qquad (v^2 = u^2 + 2fs)$$

Since the pile is inelastic, the falling mass and the pile have a common velocity immediately after impact; let v be this velocity. Total momentum after impact equals that before, therefore

$$(6 + 12)\, 112v = 6 \times 112 \times 24,$$

whence $$v = 8 \text{ ft. per sec.}$$

Let f be the retardation of the pile during its penetration. The total upward force acting upon the pile then

$$= \text{resistance of ground} - \text{total downward weight}$$

$$= 42 \times 112 \times g - 18 \times 112 \times g \text{ poundals.}$$

$$\therefore\ 112g\,(42 - 18) = 18 \times 112f, \qquad (P = mf)$$

whence $$f = \frac{24g}{18} = \frac{4g}{3}.$$

Hence, if x be the distance of penetration required

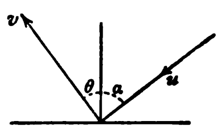
$$0 = 8^2 - 2fx, \qquad (v^2 = u^2 + 2fs)$$

$$\therefore\ x = \frac{64 \times 3}{8g} = \tfrac{3}{4} \text{ feet}$$

$$= 9 \text{ inches.}$$

148. Impact on a rough plane. *A heavy particle impinges on a given rough plane in a given direction and with a given velocity: to determine its direction and velocity after impact.*

Let m be the mass of the particle, and μ the coefficient of dynamical friction; e the coefficient of elasticity. Also let the particle impinge with velocity u at an angle α with the normal to the plane, and be reflected with velocity v at an angle θ with the normal.



By Newton's Law of Elasticity

$$v\cos\theta = eu\cos\alpha \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

The change of momentum *along the plane* (due to friction)

$$= mu\sin\alpha - mv\sin\theta.$$

The change of momentum at right angles to the plane

$$= mu\cos\alpha + mv\cos\theta,$$

$$\therefore\; mu\sin\alpha - mv\sin\theta = \mu\,(mu\cos\alpha + mv\cos\theta)$$

or $$v\,[\sin\theta + \mu\cos\theta] = u\,[\sin\alpha - \mu\cos\alpha].$$

Therefore from (1)

$$v\sin\theta + e\mu u\cos\alpha = u\,(\sin\alpha - \mu\cos\alpha),$$

i.e. $$v\sin\theta = u\sin\alpha - \mu\,(1+e)\,u\cos\alpha$$

and $$v\cos\theta = eu\cos\alpha,$$

whence, squaring and adding, we determine v; and dividing we have $\tan\theta$.

EXAMPLES. XVI*e*.

1. A body of mass 4 lbs., moving with a velocity of 22 ft. per sec., overtakes and coalesces with a body of mass 6 lbs. moving with a velocity of 12 ft. per sec.: find the velocity of this compound body after impact.

2. Two equal marbles A, B, lie in a horizontal circular groove at opposite ends of a diameter; A is projected along the groove and after a time t impinges on B: shew that a second impact takes place after a further interval $\dfrac{2t}{e}$.

3. A shell of mass 24 lbs., moving with a velocity of 1500 ft. per sec., suddenly explodes into two portions: if the smaller portion of mass 8 lbs. is reduced to rest, find the velocity of the larger.

4. A ball at rest on a smooth horizontal plane, at the distance of one yard from a wall, is impinged on directly by another equal ball moving at right angles to the wall with a velocity of one yard in a minute. If the coefficients of elasticity between the balls, and between the balls and wall, be $\frac{1}{3}$, prove that the balls will impinge a second time after 2 min. 24 secs., the radii of the balls being of inconsiderable length.

The velocity with which the falling mass strikes the pile

$$=\sqrt{2g.9}=24 \text{ ft. per sec.} \qquad (v^2=u^2+2fs)$$

Since the pile is inelastic, the falling mass and the pile have a common velocity immediately after impact; let v be this velocity. Total momentum after impact equals that before, therefore

$$(6+12)\,112v=6\times 112\times 24,$$

whence
$$v=8 \text{ ft. per sec.}$$

Let f be the retardation of the pile during its penetration. The total upward force acting upon the pile then

$$= \text{resistance of ground} - \text{total downward weight}$$

$$=42\times 112\times g-18\times 112\times g \text{ poundals.}$$

$$\therefore\ 112g\,(42-18)=18\times 112f, \qquad (P=mf)$$

whence
$$f=\frac{24g}{18}=\frac{4g}{3}.$$

Hence, if x be the distance of penetration required

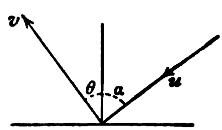
$$0=8^2-2fx, \qquad (v^2=u^2+2fs)$$

$$\therefore\ x=\frac{64\times 3}{8g}=\tfrac{6}{8} \text{ feet}$$

$$=9 \text{ inches.}$$

148. Impact on a rough plane. *A heavy particle impinges on a given rough plane in a given direction and with a given velocity: to determine its direction and velocity after impact.*

Let m be the mass of the particle, and μ the coefficient of dynamical friction; e the coefficient of elasticity. Also let the particle impinge with velocity u at an angle α with the normal to the plane, and be reflected with velocity v at an angle θ with the normal.



By Newton's Law of Elasticity

$$v \cos\theta = eu \cos\alpha \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i).}$$

The change of momentum *along the plane* (due to friction)

$$= mu \sin\alpha - mv \sin\theta.$$

The change of momentum at right angles to the plane

$$= mu \cos\alpha + mv \cos\theta,$$

$$\therefore\; mu \sin\alpha - mv \sin\theta = \mu\,(mu \cos\alpha + mv \cos\theta)$$

or $$v\,[\sin\theta + \mu\cos\theta] = u\,[\sin\alpha - \mu\cos\alpha].$$

Therefore from (1)

$$v \sin\theta + e\mu u \cos\alpha = u\,(\sin\alpha - \mu\cos\alpha),$$

i.e. $$v\sin\theta = u\sin\alpha - \mu\,(1+e)\,u\cos\alpha$$

and $$v\cos\theta = eu\cos\alpha,$$

whence, squaring and adding, we determine v; and dividing we have $\tan\theta$.

EXAMPLES. XVI*e*.

1. A body of mass 4 lbs., moving with a velocity of 22 ft. per sec., overtakes and coalesces with a body of mass 6 lbs. moving with a velocity of 12 ft. per sec.: find the velocity of this compound body after impact.

2. Two equal marbles A, B, lie in a horizontal circular groove at opposite ends of a diameter; A is projected along the groove and after a time t impinges on B: shew that a second impact takes place after a further interval $\frac{2t}{e}$.

3. A shell of mass 24 lbs., moving with a velocity of 1500 ft. per sec., suddenly explodes into two portions: if the smaller portion of mass 8 lbs. is reduced to rest, find the velocity of the larger.

4. A ball at rest on a smooth horizontal plane, at the distance of one yard from a wall, is impinged on directly by another equal ball moving at right angles to the wall with a velocity of one yard in a minute. If the coefficients of elasticity between the balls, and between the balls and wall, be $\frac{1}{3}$, prove that the balls will impinge a second time after 2 min. 24 secs., the radii of the balls being of inconsiderable length.

5. A perfectly elastic ball falls vertically for one sec. and then strikes a plane inclined at 45° to the horizon. Shew that it will again strike the plane in 2 secs. more.

6. Masses of 6 lbs. and 2 lbs. are connected by a light inextensible string over a fixed smooth pulley. The heavier mass, after moving through 8 ft. from rest, strikes a fixed inelastic horizontal plane: find the velocity with which it leaves the plane when next the string becomes taut.

7. Two equal perfectly elastic spheres are travelling in parallel directions, and one overtakes the other, the straight line joining their centres at the instant of impact making an angle θ with their direction of motion. If α and β be the angles which their directions of motion, after impact, make with the straight line joining their centres, prove that $\tan\alpha \tan\beta = \tan^2\theta$.

8. A perfectly elastic ball is projected from a point on the ground, between two parallel vertical walls, at a distance a apart, and after rebounding, first from one wall and then from the other, returns to the point of projection. If u be the velocity and α the angle of projection, prove that $u^2 \sin 2\alpha = 2ag$.

9. Compare the masses of the two portions into which a shell is divided if the velocity of one portion is reversed, and that of the other is trebled by the explosion.

10. Two small spheres, of masses $10m$ and $11m$, are projected from the same point, with equal velocities, but in opposite directions, along a circular groove. Where will the second impact take place, e being equal to $\frac{3}{4}$?

11. Three perfectly elastic balls a, b, c, of masses 1, 2, 3, are moving in the same straight line with velocities 8, 2, 1 ft. per sec. respectively; a impinges upon b, and b upon c; shew that a and b are reduced to rest, and c moves on with a velocity of 5 ft. per sec.

12. The head of a nail is struck a direct blow by a hammer of 4 lbs. mass, which at the instant of striking is moving at the rate of 20 ft. per sec. Supposing the nail to be driven into wood a distance of half an inch, and the stress between the hammer and the nail to be uniform, find its magnitude.

13. A weight P after falling through x ft. begins to raise a greater weight Q connected with it by a fine inelastic string over a smooth pulley. Shew that Q will have returned to its position after an interval

$$\frac{2P}{Q-P}\sqrt{\frac{2x}{g}}.$$

14. Two balls, of equal mass, are connected by a light inextensible string over a smooth fixed pulley, above a fixed inelastic horizontal

plane. If they are started with a velocity of 16 ft. per sec., find the velocity of each ball after three impacts with the plane have taken place.

15. A sphere, mass 16 lbs., at rest is struck by another of mass 8 lbs., moving with a velocity of 20 miles an hour in a direction making an angle of 45° with the line of centres at the moment of impact; the coefficient of elasticity being $\frac{1}{3}$. Determine the subsequent motion.

16. To a mass of 10 lbs. placed upon a smooth horizontal table a light inextensible string is attached, which passes over a small pulley at the edge of the table, and supports a mass of 2 lbs. If the smaller mass fell freely through 9 ft. before the string became tight, with what velocity would the greater mass begin to move along the table: and if the table were not smooth, and the coefficient of friction were $\frac{2}{5}$, through what distance would the greater mass move before coming to rest?

17. A series of equal, perfectly elastic, balls of different densities are arranged in the same straight line; one of them impinges on the next and so on. Prove that, if their masses form a geometrical progression of which r is the common ratio, the velocities with which they successively impinge will form a geometrical progression of which the common ratio is $\dfrac{2}{1+r}$.

18. A billiard ball A, moving parallel to one side of the table, strikes an equal ball B (initially at rest) in the centre. After B has struck the cushion and then struck A again, the velocity of A is three-fourths of its initial velocity, but in the opposite direction. Shew that, if e is the coefficient of elasticity between the balls and also between a ball and the cushion, $e^3+e^2+3e=4$.

19. If AC, BC bisect the angles between AB and the verticals through A and B respectively, and if F is the foot of the perpendicular from C on AB, the least velocity of projection which will carry a projectile from A to B is that due to a height equal to AF.

*20. A shot of m lbs. mass is fired from a gun of M lbs. mass placed on a smooth horizontal plane, and elevated at an angle α; if the direction of projection of the shot make an angle θ with the horizon, prove that $M \tan \theta = (M+m) \tan \alpha$.

21. If two perfectly elastic smooth spheres, A and B, impinge upon each other directly or obliquely, prove that B's velocity relative to A, after impact, will make the same angle with the line of centres as the relative velocity of A to B did before impact.

22. From a given point in a railway carriage, moving with uniform velocity in a straight line, bullets are fired continuously with a constant velocity at right angles to the rails, and with a constant inclination to the horizon. Find the locus at any instant of all the bullets which have not reached the ground.

23. A projectile thrown at a small elevation (3°) gives a range of 1000 yards on a horizontal plane. If the plane, instead of being horizontal, had an upward slope of 1°, what would be the range in yards, approximately?

24. To a mass $5m$ resting on a smooth horizontal table a light inelastic string is attached, which passes over a small smooth pulley at the edge of the table and supports a mass $3m$ hanging freely. After the mass $5m$ has moved 18 inches from rest a second similar string which is also attached to it becomes tight. This latter string passes over a similar pulley at the opposite edge of the table, and is attached to a mass $4m$ resting on the ground at a point vertically below the second pulley. Assuming that the horizontal portions of the strings are in a straight line, determine the additional distance through which the mass $5m$ will move before it begins to move back towards its starting point.

25. A particle of elasticity e is projected with velocity u at an angle a to the horizon, and after striking a fixed vertical wall at a horizontal distance h feet, returns to the point of projection: prove that $gh(1+e)=eu^2 \sin 2a$.

26. A solid smooth cylinder, of radius r, lies on a smooth horizontal plane, to which it is fastened, and an imperfectly elastic sphere, of radius $2r$, moves along the plane in a direction at right angles to the axis of the cylinder; prove that if the coefficient of elasticity be greater than $\frac{1}{8}$, it cannot in any case pass over the cylinder after impact.

*27. If two small perfectly elastic balls are projected at the same instant with velocities which are as $2\tan\beta$ and $(1+4\tan^2\beta)^{\frac{1}{2}}$, one up an inclined plane (angle β), another in the same vertical plane, but in a direction making an angle θ with the plane such that $2\tan\theta=\cot\beta$, prove that they will return to the point of projection at the same instant.

28. From a point at a height (h) above the ground and at the same distance from a smooth vertical wall an imperfectly elastic ball is projected against the latter with the velocity due to a fall through the height h. If $e=\frac{1}{2}$, find the direction of projection that the ball after impact may strike the ground at a distance $\frac{h}{2}$ from the foot of the wall.

29. A smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Shew that a ball of elasticity e projected along the table from a point in this rim in a direction making an angle $\tan^{-1}\sqrt{\frac{e^3}{1+e+e^2}}$ with the radius through the point will return after two impacts to the point of projection. Shew also that, when it returns to this point, its velocity will have been diminished in the ratio $1 : e^{\frac{3}{2}}$.

*30. Two spheres A and B, equal in all respects, rest in contact on a smooth table, and are impinged upon by a third sphere C of equal radius with A and B, whose centre is moving horizontally in the common tangent plane of A and B in a direction perpendicular to their common axis, the elasticity being perfect. Prove that if C's mass be half as large again as either A or B's mass, C will be reduced to rest by the impact.

31. If a perfectly elastic ball be let fall on an inclined plane, and rebound, striking the plane again, shew that the interval of the two collisions of the ball with the plane is independent of the inclination of the plane.

32. Two equal spheres A and B, lying at rest and in contact on a smooth horizontal table, are impinged upon simultaneously by a third equal sphere C. If C remain at rest after the impact, prove that the coefficient of elasticity is $\frac{2}{3}$.

33. A small elastic ball moving on a smooth horizontal table, impinges directly on an equal ball at rest. If this latter after being struck impinge directly on a vertical plane at a distance a from the point of impact, shew that the balls will be in collision again at a distance from the plane equal to $\frac{2ae^2}{1+e^2}$, e the modulus of elasticity being the same at each impact.

34. A ball whose modulus of elasticity is $\frac{1}{3}$ is let fall from a height h above an inclined plane whose elevation is 30°; after striking the plane the ball rebounds and strikes it again; shew that the range on the plane between the two points of impact is equal to $\frac{8h}{9}$.

35. An imperfectly elastic ball descends from rest from the top of an inclined plane whose height is h, and inclination 45°; shew that the range after the rebound of the ball from the horizontal plane will be h, if $\frac{1}{2}$ be the coefficient of elasticity.

36. An inelastic ball sliding along a smooth horizontal plane with a velocity of 16 ft. per second, impinges upon a smooth horizontal rail at right angles to the direction of its motion; if the height of the rail above the plane be half the radius of the ball, prove that the latus rectum of the parabola subsequently described is one foot in length.

37. A ball, whose coefficient of elasticity is e, is projected from a point in a horizontal plane so as to strike a vertical wall at a distance k, and after once rebounding on the horizontal plane returns to the point of projection: find the relation between the velocity and the direction of projection.

38. An imperfectly elastic ball is thrown from a point P so as to impinge *directly* upon a vertical wall at a point whose horizontal and

vertical distances from P are h, k. If the coefficient of elasticity be e, determine the distance of the ball from P at the instant when, after rebounding from the wall, it is vertically below P.

39. An imperfectly elastic ball moving on a smooth horizontal plane, impinges on a smooth fixed vertical plane: given the initial position of the ball, shew that, whatever be the velocity and direction of projection before impact, the line of motion of the centre of the ball after impact will pass through a fixed point.

40. Two elastic spheres, equal in all respects, are moving towards each other with equal velocities, their centres being on two parallel lines whose distance apart is d_1 (less than d, the diameter of either sphere): prove that after impact they will move away from each other with equal velocities, so that their centres are on two parallel lines whose distance apart, d_2, is given by the equation

$$d_2^2\left[e^2d^2 + (1-e^2)\,d_1^2\right] = d^2d_1^2.$$

41. A red ball is placed on a spot 6 inches from the top cushion, and 3 feet from either side cushion, of a billiard-table which is 12 ft. by 6 ft.; the coefficient of restitution between the ball and the cushion is $\frac{3}{4}$. From what point on the side cushion must another ball be aimed straight at the red one so as to drive the latter after two reflexions exactly into a corner?

149. *When two smooth bodies impinge directly, kinetic energy is dissipated or lost by the impact, unless the elasticity is perfect.*

Let m_1, m_2 be the masses of the bodies, m_1 impinging on m_2, u_1, u_2 their velocities before impact, $u_1-\alpha$, $u_2+\beta$ their velocities after impact, so that α and β are positive.

No momentum is lost,

$$\therefore\ m_1(u_1-\alpha) + m_2(u_2+\beta) = m_1u_1 + m_2u_2.$$

Whence $$m_1\alpha = m_2\beta \text{ ....(1)}.$$

By Newton's Law of Elasticity

$$(u_1-\alpha) - (u_2+\beta) = -e(u_1-u_2);$$

$$\therefore\ \alpha+\beta = (u_1-u_2)(e+1) \text{ ....(2)}.$$

Kinetic Energy before impact − that after

$$= \frac{m_1}{2}[u_1^2 - (u_1 - \alpha)^2] + \frac{m_2}{2}[u_2^2 - (u_2 + \beta)^2]$$

$$= \frac{m_1}{2}(2u_1\alpha - \alpha^2) - \frac{m_2}{2}(2u_2\beta + \beta^2)$$

$$= m_1u_1\alpha - m_2u_2\beta - \frac{m_1\alpha^2}{2} - \frac{m_2\beta^2}{2}$$

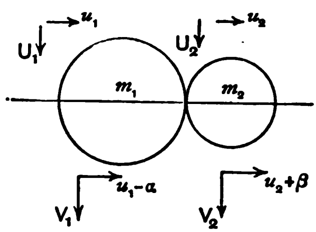
$$= m_1\alpha(u_1 - u_2) - \frac{m_1\alpha}{2}(\alpha + \beta) \qquad \text{from (1)}$$

$$= m_1\alpha(u_1 - u_2) - \frac{m_1\alpha}{2}(u_1 - u_2)(e + 1) \text{ from (2)}$$

$$= \frac{m_1\alpha}{2}(u_1 - u_2)(1 - e),$$

a positive quantity unless $e = 1$, which proves the proposition.

150. *To prove that, when two smooth bodies impinge obliquely, kinetic energy is dissipated or lost, except when the elasticity is perfect.*



Let m_1, m_2 be the masses of the bodies,

u_1, u_2 their velocities before impact along the line of impact,

U_1, U_2 " " " " perp. to " "

$u_1 - \alpha$, $u_2 + \beta$ their velocities after impact along the line of impact,

V_1, V_2 their velocities after impact perp. to the line of impact.

As in the previous article we can prove that

$$\frac{m_1}{2}[u_1^2-(u_1-\alpha)^2]+\frac{m_2}{2}[u_2^2-(u_2+\beta)^2] \ldots\ldots\ldots (1)$$

is a positive quantity except when $e = 1$, when it is zero.

Also since the effect of the impact is entirely in the line of impact,

$$V_1 = U_1 \text{ and } V_2 = U_2,$$

$$\therefore \frac{m_1}{2}[U_1^2 - V_1^2] + \frac{m_2}{2}[U_2^2 - V_2^2] = 0;$$

$\therefore$ from (1), by addition

$$\frac{m_1}{2}[(u_1^2+U_1^2)-(U_1-\alpha)^2 - V_1^2]$$

$$+\frac{m_2}{2}[(u_2^2+U_2^2)-(u_2+\beta)^2 - V_2^2]$$

is a positive quantity, except when $e = 1$.

But $\frac{m_1}{2}[u_1^2+U_1^2]$ is the K. E. of the mass m_1 before impact,

and $\frac{m_1}{2}[(u_1-\alpha)^2+V_1^2]$ „ „ „ „ after „

and so on for the mass m_2,

i.e. there is loss of Kinetic Energy except in the case when the elasticity is perfect.

This dissipated energy is expended in the generation of heat and sound.

151. *If a feet be the penetration of a shot of mass m lbs. striking a fixed iron plate horizontally with velocity v, find its penetration when the iron plate of mass M is free to move and the shot strikes it in a horizontal line with its centre of gravity.*

In the case when the plate is fixed, the work done against the shot = the kinetic energy destroyed, therefore $Pa = \frac{1}{2}mv^2$, where P poundals is the resistance to penetration.

When the plate is free to move, let V be the common velocity of the plate and shot as soon as the shot is at rest within the plate.

By Newton's 3rd Law, there is no momentum lost,

$$\therefore\ (m+M)V = mv \ \dots\dots\dots\dots\dots\dots (1).$$

Also if x ft. be the distance of penetration here, the work done against the shot $=$ the change in the kinetic energy of the system,

$$\therefore\ Px = \tfrac{1}{2}mv^2 - \tfrac{1}{2}(m+M)V^2,$$

$$\therefore\ \frac{mv^2}{2a}x = \tfrac{1}{2}mv^2 - \tfrac{1}{2}\frac{m^2v^2}{m+M} \text{ from (1),}$$

and hence
$$x = \frac{aM}{M+m} \text{ feet.}$$

152. *A heavy perfectly flexible uniform string hangs with its lowest end at a height h above an inelastic horizontal plane. If the string be allowed to fall, find the pressure on the table at any time during the motion.*

Let m be the mass of unit length of the string.

There will be no tension in any portion of the string during its fall.

This can be seen by considering any portion of the string: take a length l measured from the lowest end, and let T be the tension at the upper end of this portion.

The moving force acting upon this $= mg - T$ vertically downwards,

$$\therefore\ mg - T = mg, \qquad (P = mf)$$

$$\therefore\ T = 0.$$

Hence each particle of the string will fall freely and independently under the action of gravity.

At time t the velocity of each particle $= gt$; therefore if the string has reached the table, the mass of string which

reaches the table per sec. $= mgt$; therefore momentum per second destroyed by the table

$$= mgt\,.\,(gt) = mg^2t^2.$$

Also at this instant there will be lying on the table a mass of length $\frac{1}{2}gt^2 - h$; and the pressure on the table of this portion

$$= mg\left(\tfrac{1}{2}gt^2 - h\right).$$

Therefore total pressure at time t on the table = momentum destroyed per sec. by the table + weight of portion on the table

$$= mg^2t^2 + mg\left(\tfrac{1}{2}gt^2 - h\right) = \tfrac{3}{2}mg^2t^2 - mgh.$$

EXAMPLES. XVI *f*.

1. An inelastic ball slides down a smooth inclined plane AB of elevation 30°, through a distance of 8 feet, and then strikes a horizontal plane BC. Prove that the ball runs along the plane BC with a velocity of $8\sqrt{3}$ ft. per second. After a time the ball reaches a rough part of the plane BC where the coefficient of friction is $\frac{1}{4}$. How much further will it run?

2. AB, BC, CD are smooth planes, BC being horizontal, and AB, CD inclined at an angle of 30° to the horizon. An inelastic ball slides down AB through a distance of 2 feet. What is its velocity along BC, and how far will it ascend CD?

3. An inelastic ball, after sliding down an inclined plane of elevation a, strikes a rough horizontal plane, coefficient of friction μ, with velocity u. Prove that the ball starts along this plane with velocity $u\cos a - \mu u \sin a$.

4. Two balls lying on a smooth table are connected by a light inelastic string. If the balls are of equal mass, m, and one is projected along the table with velocity u, find the loss of kinetic energy due to the jerk of the string when it tightens.

5. A pile weighing 4 cwt. is being driven into the ground by a weight of 3 cwt. let fall on it from a height of 13 feet. If each blow drives the pile $\frac{3}{4}$ of an inch, show that the resistance of the earth to penetration is between 13 and 14 tons wt. (Assume this resistance to be a constant force and that after the blow, the weight and the pile move on together until stopped by this force.)

6. Two balls, of masses 4 lbs. and 2 lbs., moving in opposite directions with velocities 20 and 10 ft. per sec. respectively, impinge directly. Find in ft.-lbs. the kinetic energy dissipated by the impact, if the coefficient of elasticity be $\frac{1}{3}$.

7. A smooth ball strikes a ball of equal mass lying at rest. If the moving ball makes before impact an angle of 30° with the line joining the centres at the instant of impact, find the total loss of kinetic energy caused by the impact.

8. A perfectly inelastic body, of given mass and moving with a given velocity, is brought to rest by coming into direct collision with a second perfectly inelastic body moving in the opposite direction. Show that the kinetic energy of the second body must be inversely proportional to its mass.

9. Prove that the resistance of the wood is 204 lbs. wt. to a nail weighing 1 oz., supposing a hammer weighing 1 lb. striking with a horizontal velocity of 34 ft. per sec. drives the nail 1 inch into a fixed block of wood.

If the block is free to move, and weighs 68 lbs., prove that the hammer will drive the nail only $\frac{64}{65}$ inch.

10. A gun weighing 4 tons, fires a shot of 160 lbs. If the powder generates 10^7 ft.-lbs. of mechanical energy, determine the muzzle velocity of the shot to the nearest 10 ft. per sec. Assume the gun horizontal and free to recoil.

11. A railway truck of mass m when moving with a velocity $3v$ strikes a truck of mass m' at rest, and moves after impact with a velocity v in the same direction as before. Determine the velocity of the second truck after impact, and the kinetic energy lost by the system.

12. A force of 1200 tons wt. acts upon a shot weighing 700 lbs. whilst it moves through a distance of 12 feet in the direction of the force, and then ceases. Supposing the shot without diminution of velocity to strike a target and penetrate it to a depth of 12 inches, then coming to rest, find the pressure exerted by the shot on the target, supposing it to be uniform during penetration.

13. A man of mass M standing on perfectly smooth ice picks up a stone of mass m as it is sliding towards him with velocity v. At what rate will the man begin to slide? State precisely what he must do with the stone if he wishes to return to his original station at the same speed as that with which he left it.

14. In the previous question how could the man bring himself to rest again?

15. A bullet weighing 50 grammes is fired into a target with a velocity of 500 metres per sec. The target weighs a kilogramme, and is free to move. Find in kilogram-metres the loss of energy in the impact.

16. A gun is fired when it is moving forwards horizontally with a velocity of 6 ft. per sec., and the recoil brings it to rest. The mass of the gun and carriage is 100 tons, of the shot 1000 lbs., and the mass of the powder may be neglected. Find the velocity with which the shot leaves the gun.

17. A cannon ball, whose mass is 1 cwt., strikes a suspended target of mass 10 tons with a velocity of 1200 ft. per sec. Find the velocity with which the target begins to move after the impact (1) when the shot is embedded in it, (2) when the shot just penetrates it.

18. A 2 oz. bullet, striking a fixed block, penetrates it to a depth of 6 inches; prove that if the block, of mass 10 lbs., be free to move, the bullet will penetrate to a depth of $\frac{40}{81}$ feet, supposing the bullet to strike the block in a horizontal line through the C.G. of the block.

19. A smooth tube is bent into the form Γ, so as to have a sharp corner. Two particles B and C joined to a third particle A by separate strings of lengths b feet, $b+c$ feet respectively are placed in the horizontal tube. Initially A is placed at the top of the vertical tube, while B and C are as far from A as the strings permit. The three particles being of equal mass, find how many seconds will have elapsed before B reaches the top of the vertical tube. Show that if $3c$ is not greater than $2b$ then c will come to the top of the vertical tube $\frac{(2b+c)\sqrt{3}}{\sqrt{2bg}}$ seconds after the motion begins.

20. Two bodies whose masses are m and m' are connected by a cord passing over a smooth pulley. The pulley is at a height $2h$ above a table, on which is another body of mass $m-m'$, vertically under m', and attached to m' by another cord of length h. The system is let go from a position in which m' is at a height $\frac{h}{2}$ above the table, and the table does not interfere with the motion of m. Find the velocity of m' just before the second cord becomes tight.

Find also the velocity of m just after the second cord becomes tight.

21. Two scale pans of mass $2m$ and m are connected by a light inelastic string and hang over a smooth pulley. When the system is moving with velocity v, a ball of mass m falling at the instant with velocity v drops into the rising pan and sticks to it. Find the velocity of the system immediately after the impact.

22. In the previous problem find the velocities immediately after impact of the various masses if e is the coefficient of elasticity of the falling ball.

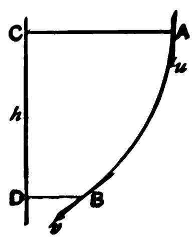
CHAPTER XVII.

MOTION ON A SMOOTH CURVE UNDER THE ACTION OF GRAVITY.

153. *If a particle, having initial velocity u, slide down an arc of a smooth curve in a vertical plane, when it has descended through a vertical height h, its velocity (v) will be given by the formula:—*

$$v^2 = u^2 + 2gh.$$

1*st Proof; based on the Principle of the Conservation of Energy.*



The direction of motion at any point on the curve is along the tangent at that point. The reaction of the curve is perpendicular to this direction.

Therefore the reaction of the curve does no work on the body. Hence the only work done is that done by gravity.

Let m be the mass of the particle.

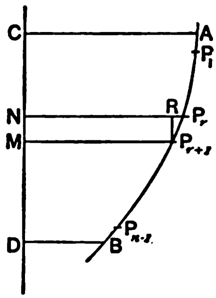
The change of its kinetic energy = the work done by gravity.

$$\therefore \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = mg \, . \, h.$$

$$\therefore v^2 = u^2 + 2gh.$$

Alternative Proof.

Let the particle start down the arc AB from A with velocity u, and let v be its velocity when it reaches B, a point at a vertical distance h below A. Suppose the arc divided into a *large number* (n) of equal arcs, so that each portion is *very small*, and therefore approximately a straight line.



Let P_1, $P_2 \ldots P_{n-1}$ be the points of division, and let v_1, $v_2 \ldots$ be the velocities of the particle at these points respectively. Consider the rth division P_rP_{r+1}. Draw P_rN and $P_{r+1}M$ horizontally and $P_{r+1}R$ vertically to meet P_rN at R; then $P_{r+1}R = \frac{h}{n}$. P_rP_{r+1} being approximately a straight line, the acceleration along it is $g \cos P_rP_{r+1}R$.

$$\therefore \; v^2_{r+1} = v_r^2 + 2g \cos P_rP_{r+1}R \times P_rP_{r+1}$$

$$= v_r^2 + 2g \, . \, P_{r+1}R$$

$$= v_r^2 + 2g\frac{h}{n}.$$

Similarly at P_1 $\quad v_1^2 = u^2 + 2g\frac{h}{n}$, and so on.

$$\therefore \; v_2^2 = v_1^2 + 2g\frac{h}{n},$$

....

....

$$v^2_{n-1} = v^2_{n-2} + 2g\frac{h}{n},$$

$$v^2 = v^2_{n-1} + 2g\frac{h}{n};$$

therefore by addition $\quad v^2 = u^2 + 2gh.$

COR. If the body be projected *up* the curve with velocity u, instead of downwards, we have in the same manner

$$v^2 = u^2 - 2gh,$$

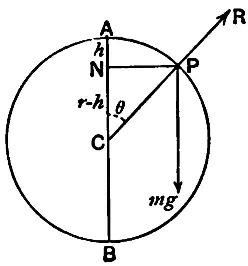
where v is its velocity when it is at a vertical distance h above its point of projection.

154. We notice that the vertical height through which the particle moves is independent of the *shape* of the curve. The particle may first ascend, then descend, ascend again, and so on. The *time*, of ascent or descent as the case may be, cannot in general be found without the help of the Differential Calculus.

The theorem can also be applied to the motion of a particle at the end of an inelastic string fastened to a fixed point; for the constraining force, the tension of the string, will always be normal to the path of the particle.

155. *A particle slides down the outside of a smooth vertical circle. To find where it leaves the curve, if it starts from rest at the highest point.*

Let AB be the vertical diameter of the circle, C its centre, r its radius.



Let P be the position of the particle, of mass m, at any instant, v its velocity at that point.

Draw PN horizontally to meet AB at N. And let $AN = h$, angle $ACP = \theta$.

The forces acting on the particle are:—

(1) R the normal pressure of the curve along the radius CP.

(2) mg the weight of the particle vertically downwards.

The total force in direction PC is, therefore,

$$mg \cos\theta - R.$$

But since the body is describing a circle, the resultant force must be $\frac{mv^2}{r}$ along PC. (Art. 95)

$$\therefore\ mg\cos\theta - R = \frac{mv^2}{r} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Also since the body slides under the action of gravity on a smooth curve

$$v^2 = 2g\,.\,AN = 2gh. \qquad \text{(Art. 153)}$$

Hence from (1)
$$R = m\left[g\cos\theta - \frac{v^2}{r}\right]$$
$$= m\left[g\left(\frac{r-h}{r}\right) - \frac{2gh}{r}\right]$$
$$= mg\left(\frac{r-3h}{r}\right).$$

Now the reaction R acts outwards, and therefore the particle will remain on the curve as long as R is positive; and at the instant when it leaves the curve $R = 0$.

Hence if P be the point where it leaves the curve

$$mg\left(\frac{r-3h}{r}\right) = 0,$$

and
$$h = \frac{r}{3}.$$

Its velocity at that instant $= \sqrt{2gh} = \sqrt{\frac{2gr}{3}}$.

After leaving the curve the particle describes a parabola.

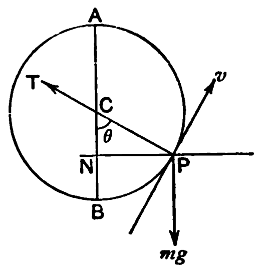
156. *A particle, of mass m, is suspended by a string, of length r, from a fixed point. It is then projected with velocity u in a horizontal direction so that it describes a vertical circle: to find*

(1) *the velocity of the particle at any point in its path;*

(2) *the tension of the string " " " ";*

(3) *the initial velocity if the particle just makes complete revolutions.*

Let ACB be the vertical diameter of the circle, v the velocity of the particle when it is at P, T the tension of the string, θ the angle PCB, h the vertical height of P above B the lowest point.



Draw PN perpendicular to AB, so that $BN = h$.

Then

$$v^2 = u^2 - 2gh \ldots\ldots \text{(i)},$$

$$\therefore\ v = \sqrt{u^2 - 2gh}.$$

Also since the particle is moving in a circle, the resultant force along the normal $PC = \dfrac{mv^2}{r}$. (Art. 95)

$$\therefore\ T - mg\cos\theta = \frac{mv^2}{r}.$$

$$\therefore\ T = mg\cos\theta + \frac{mv^2}{r}$$

$$= m\left[g\left(\frac{r-h}{r}\right) + \frac{u^2 - 2gh}{r}\right], \text{ from (i)},$$

$$= \frac{m}{r}\left[u^2 + g(r - 3h)\right] \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

Thus we have found the velocity (v) and the tension (T) at any point.

Now T must remain positive throughout the motion, for a string cannot exert a thrust; also from equation (ii) we see that T is least when h is greatest, i.e. at the highest point.

Therefore the particle will just make complete revolutions if the tension at the highest point is zero.

In this case, we have from equation (ii)

$$u^2 + g(r - 6r) = 0.$$

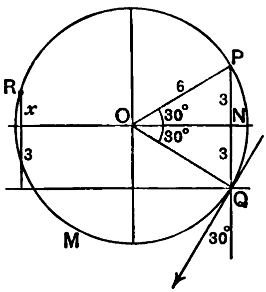
$$\therefore\ u^2 = 5gr,$$

$$u = \sqrt{5gr}.$$

Therefore the least velocity the particle can have at its lowest point, if it is to make complete revolutions, is $\sqrt{5gr}$.

157. *A ball is connected with a fixed point O by means of an inelastic string of length 6 feet. It is held, with the string taut, at a point 3 feet above the horizontal through O, and then let fall: find its velocity immediately after the string becomes taut again, and the height, above O, to which it subsequently rises.*

Let the body drop from P, so that the string becomes taut again when the body reaches Q; we see that $PQ = 6$ feet.



Let v be the velocity of the body at Q,

$$v^2 = 2g \times 6 = 12 \times 32 \ldots \text{(i)}.$$

At Q the impulsive tension of the string destroys the component of Q's velocity in the direction QO; and the velocity, $v \cos 30°$, at right angles to the string is unaffected by the impulse. The velocity of the ball immediately after the string becomes taut is therefore $v \cos 30° = v\dfrac{\sqrt{3}}{2}$ along the tangent at Q. The ball then describes the circle QMR, and the motion is equivalent to that on a smooth curve.

Let the body rise to a height x above the horizontal through O, then we have a velocity $v \cos 30°$ destroyed by gravity in the vertical height $(x + 3)$ feet.

$$0 = (v \cos 30°)^2 - 2g(x + 3), \qquad (v^2 = u^2 + 2fs)$$

$$2g(x + 3) = 12 \times 32 \times \tfrac{3}{4}, \text{ from (i)},$$

$$2x + 6 = 9,$$

$$x = 1\tfrac{1}{2} \text{ feet}.$$

EXAMPLES. XVII.

1. A particle of mass 3 lbs. is attached to a fixed point by means of a string 4 ft. long; it is held with the string horizontal and then let fall: find the tension of the string and the velocity of the particle when it reaches its lowest point.

2. A particle of mass 8 lbs. hangs at one end of a string 4 ft. long, the other end being attached to a fixed point, and is projected horizontally with a velocity of 32 ft. per sec.: find the tension of the string when the particle has risen through a vertical distance of 6 feet.

3. A heavy particle slides from rest at the highest point down the inside of a hemispherical bowl placed with its rim horizontal. Shew that the velocity at the lowest point varies as the square root of the radius of the bowl.

4. A bead slides on a wire bent into the form of a parabola whose axis is vertical and vertex upwards; if the bead be just displaced from its position of equilibrium, then at any subsequent time its velocity will vary as its distance from the axis.

5. A ring loosely strung on a smooth vertical circular wire slides from rest at any point: shew that its velocity at the lowest point varies as the chord of the arc of descent.

6. One end of a string is fixed, and a weight of 1 lb. is attached to the other. If the weight be raised till the string is horizontal, and then let go, find the tension of the string when it becomes vertical.

If the string is just able to bear a strain of 1 lb., find at what point of the fall it will break.

7. If a heavy particle is allowed to fall from the edge down the interior of a smooth hemispherical bowl, prove that its pressure on the surface at the lowest point is three times the weight of the particle.

8. A heavy particle is tied to the end of a string 10 feet long, the other end of which is fastened to a point A; at a distance 3 ft. below A, and in the same vertical line with it, is a peg B; the particle descends through an angle of 45°, when the string comes to the peg B; find the angle through which the particle will rise afterwards.

9. A small heavy ring can slide upon a cord 34 ft. long which has its ends attached to two fixed points A, B in the same horizontal line and 30 ft. apart. The ring starts—the string being tight—from a point 5 ft. from A; shew that, when it has described a length of the cord equal to 3 ft., its velocity will be 10·12 ft. per sec. nearly.

10. A heavy particle is connected by an inextensible string 3 ft. long to a fixed point, and describes a circle in a vertical plane about that point, its velocity at its lowest point being that due to a fall through 10 ft.; find the ratios of the tensions of the string at the

highest and lowest points of the circle, and when the string is horizontal.

11. A heavy ring slides down a smooth parabola whose axis is vertical and latus rectum $2\frac{1}{4}$ ft., starting from rest at the vertex: find its velocity when it is 5 ft. from the starting point.

12. A particle slides down an arc of a vertical circle from rest at one extremity of the horizontal diameter: find expressions for the acceleration along and perpendicular to the arc and for the velocity and for the pressure on the arc at any point.

13. A weight P is attached to the extremity of an inextensible string OP fixed at the end O. P is held so that OP is stretched at an angle a to the vertical and above O. If P be then allowed to fall, prove that when it next comes to rest its height above O will be to its initial height as $\cos(\pi - 2a) : 1$.

14. A cannon weighing 12 cwts. swinging horizontally by two vertical suspending ropes of equal length, 9 feet, projects a ball and is raised by the recoil 2·25 ft. above its lowest position. Find the momentum of the ball, and the tension of the ropes at the instant of discharge. Find also the tension of the ropes at the instant when the cannon reaches its highest point.

15. A heavy bead, loosely strung on a smooth vertical circular wire, falls down it from rest at the highest point O. When at any assigned point, find the rate at which its distance from O (in a straight line) is increasing.

16. A heavy particle slides from rest at the highest point of the surface of a smooth fixed sphere of radius r. Find the latus rectum of the parabola described by the particle after leaving the sphere.

17. A heavy particle is projected horizontally, with velocity u, from the highest point of a vertical circle of radius r: find where it leaves the curve.

18. An elliptical wire of eccentricity $\sqrt{\frac{2}{3}}$ is placed in a vertical plane with its major axis inclined to the horizontal at an angle 60°. If a small heavy ring is allowed to slide along the wire from rest at the higher extremity of the major axis, shew that its velocities at the two ends of the minor axis are as $\sqrt{2} : 1$.

19. A system of circles in the same vertical plane is drawn having a common highest point. Particles slide down the circles from rest at the highest point, and after a time leave the circles. Shew that the foci of the free paths all lie on a straight line, and find its inclination to the vertical.

20. A heavy particle slides from rest at a point on a smooth fixed sphere, of radius r, at an angular distance a from its highest point. Shew that the latus rectum of the parabola which the particle describes after leaving the sphere is $\dfrac{16r\cos^3 a}{27}$.

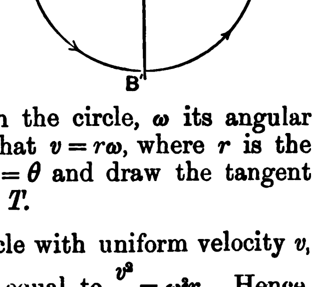
CHAPTER XVIII.

SIMPLE HARMONIC MOTION. PENDULUM.

158. Simple Harmonic Motion.

If a point move in a straight line so that its acceleration is always directed towards a fixed point in that line, and varies directly as its distance from that point, then the moving point is said to have a simple harmonic motion.

A point P starting from A describes the circle $ABA'B'$ with uniform velocity; take the diameter ACA' of this circle and draw PQ perpendicular to it; to prove that while P describes the circle uniformly, the point Q has a simple harmonic motion.



We have to shew that the acceleration of Q is towards the centre C, and varies as CQ.

Let v be the velocity of P in the circle, ω its angular velocity about the centre C, so that $v = r\omega$, where r is the radius of the circle. Let $\angle PCQ = \theta$ and draw the tangent PT at P to meet CA produced at T.

Now since P describes the circle with uniform velocity v, its acceleration is along PC, and equal to $\frac{v^2}{r} = \omega^2 r$. Hence,

since PQ is always at right angles to AA', and Q moves along AA', the acceleration of Q is equal to the component of P's acceleration in the direction QC;

i.e. Acceleration of Q

$$= \omega^2 r \cos PCQ$$

$$= \omega^2 r \cos \theta$$

$$= \omega^2 r \cdot \frac{CQ}{CP}$$

$$= \omega^2 x, \text{ where } CQ = x,$$

$$\propto x, \text{ since } \omega \text{ is constant.}$$

Thus we have shewn that the acceleration of Q is towards the centre C, and varies directly as its distance from C.

Therefore the point Q moves with a simple harmonic motion.

159. *To find the velocity of the point Q, and to shew that it oscillates from A to A', and back to A.*

The velocity of P

$$= v = r\omega \text{ along } TP,$$

and the velocity of Q

$$= \text{the component of } P\text{'s velocity along } QC$$

$$= r\omega \cos PTC$$

$$= r\omega \sin \theta$$

$$= r\omega \cdot \frac{PQ}{CP}$$

$$= \omega \cdot PQ$$

$$= \omega \sqrt{r^2 - x^2}.$$

We see that as x diminishes the velocity of Q increases, and vice versâ.

When P is at A the velocity of $Q = 0$, for $x = r$.

As P moves from A to B the velocity of Q increases, for x diminishes.

When P is at B the velocity of $Q = \omega r$, for $x = 0$.

As P moves from B to A' the velocity of Q decreases, for x increases.

When P is at A' the velocity of $Q = 0$, for $x = r$.

As P moves from A' to B' the velocity of Q is reversed (for the acceleration is towards C always) and increases.

When P is at B' the velocity of $Q = \omega r$, for $x = 0$.

As P moves from B' to A' the velocity of Q diminishes, for x increases.

When P is again at A' the velocity of Q again $= 0$.

After this the motion repeats itself.

We have thus shewn that the motion of Q is oscillatory from A to A' and back, that Q's velocity is zero at A and A' and greatest ($= r\omega$) at the centre C.

160. *To find the time of a complete oscillation from A to A', and back to A.*

The time taken by Q to move from A to Q

$=$ the time taken by P to move in the circle from A to P

$= \dfrac{\theta}{\omega}$, for ω is the angular velocity of P in the circle;

therefore the time from A to $A' = \dfrac{\pi}{\omega}$,

and the time of a complete oscillation from A to A' and back to A

$$= \frac{2\pi}{\omega}.$$

Cor. The time from A to P may be expressed in terms of x, for

$$\cos\theta = \frac{x}{r}.$$

Hence, time from A to P

$$=\frac{\theta}{\omega}=\frac{1}{\omega}\cos^{-1}\left(\frac{x}{r}\right).$$

161. In cases of Simple Harmonic Motion we usually take $\omega^2=\mu$.

This makes the acceleration $\mu \,.\, CQ$ (Art. 158)

$$=\mu x.$$

Amplitude. The range CA or CA' of the moving point on either side of the centre C is called the *Amplitude* of the oscillation.

Periodic Time. The interval of time from the instant when the moving point leaves one position to the instant that it reaches the same position again, having the same velocity and direction as before, is called the **Periodic Time.**

It will be seen from the preceding articles that we have established the following, for **Simple Harmonic Motion:**

Acceleration towards the fixed point $C=\mu x$.

Velocity at distance x from the fixed point $C=\sqrt{\mu\,(r^2-x^2)}$.

The velocity at A and also at A' is zero.

Maximum velocity $=r\sqrt{\mu}$ at C.

Time from rest to a point Q distant x from C

$$=\frac{\theta}{\sqrt{\mu}}=\frac{1}{\sqrt{\mu}}\cos^{-1}\left(\frac{x}{r}\right).$$

Periodic Time $=\dfrac{2\pi}{\sqrt{\mu}}$.

We also observe that the Periodic Time is independent of the Amplitude; i.e. whether the oscillation be great or small, the periodic time is the same.

Such an oscillation or vibration is said to be isochronous.

PENDULUMS.

162. Simple Pendulum. A small heavy mass, termed he *bob*, suspended from a fixed point by means of a light tring or wire, and made to oscillate in a vertical plane, is alled a Simple Pendulum.

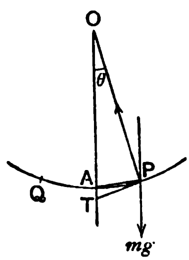
The time of swinging from rest to rest is called its *Time f Oscillation or Vibration*, or a *beat.*

The time of swinging from rest to rest, and *back again to est* (often termed a *swing-swang*), is called its *Periodic Time.*

We shall only be concerned in this chapter with the scillations of simple pendulums of very small amplitude, nd by 'small' we mean that the ratio of the arc of oscillation o the length of the pendulum is small.

163. *If l be the length of a simple pendulum oscillating hrough a small angle, the time of a complete oscillation swing-swang) is* $2\pi\sqrt{\dfrac{l}{g}}$.

Let O be the point of suspension, m the mass of the bob, PQ the arc of oscillation. Draw OA vertically so that A is the lowest point of he arc.



Draw PT the tangent at P to meet OA produced at T, and let $\angle AOP = \theta$.

Since the arc AP is *very small,* the hord AP, the arc AP, and the tangent PT are approximately equal and coincident; also the angles OPA, OAP are *very nearly* right angles.

The only forces acting on the bob are the tension of the tring along PO, and the weight of the bob.

Hence the only force in the direction of motion of the bob, i.e. along PA, is the component of mg in that direction,

$$= mg \sin \theta$$

$$= mg\theta, \text{ approximately}$$

$$= \frac{mg \,.\, AP}{l}.$$

Therefore the mass m has an acceleration towards A equal to

$$\frac{g \,.\, AP}{l}.$$

This varies as AP, the distance from A.

Therefore we have a simple harmonic motion.

Therefore the Period of oscillation (time of swing-swang)

$$= \frac{2\pi}{\sqrt{\frac{g}{l}}} \text{ seconds} \qquad \left(\frac{2\pi}{\sqrt{\mu}}. \text{ Art. 160}\right)$$

$$= 2\pi\sqrt{\frac{l}{g}} \text{ seconds.}$$

Thus we see that provided the amplitude **is small, the** Period of oscillation of a simple pendulum is **isochronous,** and therefore independent of the amplitude.

Also, under the same condition, the period **of oscillation** varies directly as the square root of the length **of the** pendulum.

164. Simple Equivalent Pendulum.

In the previous article we have neglected the **mass** of the string or wire, and we have considered the **bob as a** particle; but with an ordinary pendulum, such **as that used** with a clock, the bob is of appreciable **size,** and the **wire** of appreciable mass.

The length of a *simple equivalent pendulum* is the **length** of that simple pendulum which has an equal **periodic time.**

165. The Seconds Pendulum.

To find the length of a seconds pendulum, i.e. *of a pendulum which makes a swing from rest to rest in one second.*

[N.B. Its time of complete oscillation = 2 secs.]

Let l be its length, then by Article 163,

$$1 = \pi\sqrt{\frac{l}{g}}.$$

$$\therefore\ l = \frac{g}{\pi^2} \text{ feet.}$$

The length therefore depends on the value of g, the acceleration of gravity.

Taking $g = 32$ and $\pi = \frac{22}{7}$

$$l = 38{\cdot}88 \text{ inches.}$$

166. *To find the change in the number of oscillations, or vibrations, in a given time due to a change of length of pendulum.*

Let a pendulum of length l make n beats in t seconds and a pendulum of length $l + a$ make $n - x$ beats in t seconds.

In the first case, time of a beat

$$= \frac{t}{n} = \pi\sqrt{\frac{l}{g}}. \qquad \text{(Art. 163.)}$$

In the second case, time of a beat

$$= \frac{t}{n-x} = \pi\sqrt{\frac{l+a}{g}},$$

$$\therefore\ \frac{n-x}{n} = \sqrt{\frac{l}{l+a}},$$

$$\therefore x = n\left[1 - \sqrt{\frac{l}{l+a}}\right] = n\left[1 - \left(1 + \frac{a}{l}\right)^{-\frac{1}{2}}\right].$$

COR. If a be small compared with l,

$$x = n\left[1 - \left(1 - \frac{a}{2l} + \dots\right)\right] \quad \text{[Binomial Theorem]}$$

$$= \frac{na}{2l} \text{ approximately.}$$

167. *To find the change in the number of oscillations in a given time due to a change in the value of g.*

Let a pendulum of length l make n beats in time t, with acceleration g, and let it make $n+x$ beats in time t, with acceleration $g+g'$.

In the first case, the time of one beat

$$= \frac{t}{n} = \pi\sqrt{\frac{l}{g}}. \qquad \text{(Art. 163.)}$$

In the second case, the time of one beat

$$= \frac{t}{n+x} = \pi\sqrt{\frac{l}{g+g'}}, \qquad \text{(Art. 163.)}$$

$$\therefore \frac{n+x}{n} = \sqrt{\frac{g+g'}{g}},$$

$$\therefore x = n\left[\sqrt{\frac{g+g'}{g}} - 1\right].$$

The value of g does not vary much on the earth's surface, hence in that case g' is small, and powers of $\frac{g'}{g}$ may be omitted.

$$\therefore x = n\left[\left(1 + \frac{g'}{g}\right)^{\frac{1}{2}} - 1\right]$$

$$= n\left[1 + \tfrac{1}{2}\frac{g'}{g} - 1\right] \text{ approximately}$$

[Binomial Theorem]

$$= \frac{ng'}{2g} \text{ approximately.}$$

168. Variation in the value of g.

Every particle attracts every other particle with a force which varies directly as the product of the masses, and inversely as the square of the distance between them.

Newton discovered this law, which holds throughout the universe. From this it can be proved (the proof does not come within the range of Elementary Dynamics) that the attraction of a sphere on any mass *outside* it is the same as if the whole mass of the sphere were collected at its centre, i.e. as if the sphere were a particle.

Hence the attraction of the earth on any particle outside it varies inversely as the square of the distance from the centre.

Therefore if g be the value of gravity at the earth's surface, g_1 the value of gravity at height h from the earth's surface, and r the radius of the earth,

$$\frac{g_1}{g} = \frac{r^2}{(r+h)^2}.$$

Similarly it can be proved that

The attraction on a particle **inside** *the earth varies directly as its distance from the centre.*

Therefore if g_2 be the value of gravity at a depth d below the surface of the earth,

$$\frac{g_2}{g} = \frac{r-d}{r}.$$

169. *To find the change in the number of oscillations or beats, in a given time, due to ascending to a height h above the surface of the earth.*

Let r be the radius of the earth.

Let a pendulum of length l make n beats in time t, with acceleration g, at the surface of the earth, and $n-x$ beats in time t, with acceleration $g-g'$, at height h from the earth.

In the first case, the time of one beat

$$= \frac{t}{n} = \pi\sqrt{\frac{l}{g}}. \qquad \text{(Art. 163.)}$$

In the second case, the time of one beat

$$=\frac{t}{n-x}=\pi\sqrt{\frac{l}{g-g'}},$$

$$\therefore \frac{n-x}{n}=\sqrt{\frac{g-g'}{g}},$$

$$=\frac{r}{r+h} \text{ (Art. 168)}$$

i.e.
$$1-\frac{x}{n}=\frac{1}{1+\frac{h}{r}}$$

$$=\left(1+\frac{h}{r}\right)^{-1}$$

$$=1-\frac{h}{r} \text{ approximately (Binomial Theorem),}$$

for h is small compared with r.

$$\therefore x=\frac{nh}{r},$$

i.e. the decrease in the number of beats $=\frac{nh}{r}$.

170. *To find the change in the number of oscillations or beats, in a given time, due to descending to a depth* **d** *below the surface of the earth.*

Let r be the radius of the earth.

Let a pendulum of length l make n beats in time t, with acceleration g, at the surface of the earth, and $n-x$ beats in time t, with acceleration $g-g'$, at the depth d.

In the first case, the time of one beat

$$=\frac{t}{n}=\pi\sqrt{\frac{l}{g}}.$$

In the second case, the time of one beat

$$=\frac{t}{n-x}=\pi\sqrt{\frac{l}{g-g'}}.$$

$$\therefore \frac{n-x}{n}=\sqrt{\frac{g-g'}{g}}$$

$$=\sqrt{\frac{r-d}{r}} \text{ (Art. 168).}$$

$$\therefore 1-\frac{x}{n}=\left(1-\frac{d}{r}\right)^{\frac{1}{2}}$$

$$=1-\frac{d}{2r} \text{ approx. (Binomial Theorem)}$$

for d is small compared with r.

$$\therefore x=\frac{nd}{2r},$$

i.e. the decrease in the number of beats

$$=\frac{nd}{2r}.$$

EXAMPLES. XVIII.

[$\pi = \frac{22}{7}$. Radius of earth = 4000 miles.]

1. A particle moves with simple harmonic motion, starting from a point 5 ft. from the centre of its path: find its maximum velocity when 11 secs. is its periodic time.

2. A particle moving with simple harmonic motion has an acceleration of 4 ft.-sec. units at a distance of 1 foot from the centre of its path: find its amplitude when its velocity at a distance of 3 ft. from its centre is 4 ft. per second.

3. A particle moving with simple harmonic motion has a velocity of 4 ft. per sec. when at a distance of 1 ft. from its centre, and a velocity of 3 ft. per sec. when at a distance 2 ft.: prove that its periodic time is $\frac{2\pi\sqrt{21}}{7}$ seconds.

4. A particle moving with simple harmonic motion starts at a point 7 ft. from the centre of its path and has a maximum velocity of 11 ft. per sec.: find its periodic time.

5. Find the length of a pendulum whose beat is 5 seconds.

6. Find the beat of a pendulum 800 ft. long.

7. How many oscillations will a pendulum 8 feet long make in a day?

8. How many beats will a pendulum 17·44 centimetres long make in 352 secs.? ($g = 981$.)

9. A clock with a seconds pendulum loses 8 secs. a day: find approximately the required alteration in its length.

10. A pendulum 3 ft. long is observed to make 700 oscillations in 671 secs.; find approximately the value of g.

11. A seconds pendulum is lengthened by 2 inches: find how many seconds it then loses in an hour.

12. A pendulum which beats seconds where $g=32$ is taken to a place where $g=32{\cdot}2$: find the number of seconds it gains in 4 hours.

13. A pendulum beats seconds at sea-level: find approximately the height above sea-level of the summit of a mountain where the same pendulum beats 3599 times in an hour.

14. A pendulum which beats seconds at the surface of the earth is taken down a mine half a mile deep: how many oscillations does it then lose in a day?

15. A simple pendulum performs 21 complete vibrations in 44 secs.; on shortening its length by 47·6875 centimetres, it performs 21 complete vibrations in 33 secs.: from this determine the value of gravity in cm.-sec. units.

16. A clock is regulated by a pendulum which would beat true seconds at the pole, but such that the clock would lose 9 secs. in the hour if at the equator. In the latter situation the clock will however keep correct time if the pendulum be shortened by $\frac{8}{41}$ths of an inch. Find the values of gravity at the pole and at the equator.

17. A pendulum whose length is l makes m oscillations in one day; its length is diminished by a small quantity and it is found to make $m+n$ oscillations in a day: shew that the diminution of its length is equal to $\frac{2ln}{m}$ nearly.

18. A seconds pendulum is lengthened ·05 inch: find the number of seconds it will lose in 24 hours.

19. A pendulum of length l has one end of the string fastened to a peg on a smooth plane inclined to the horizon at angle a. With the string and the weight on the plane its time of oscillation is 2 secs. Find a, having given that a pendulum of length $\frac{l}{2\sqrt{2}}$ oscillates in one second when suspended vertically.

20. A clock which at the surface of the earth at a certain place gains 10″ a day, loses 10″ a day when taken down a mine; compare the force of gravity at the surface and at the bottom of the mine.

21. A seconds pendulum loses 20 beats per day when taken to the top of a mountain. Find the height of the mountain.

22. If a seconds pendulum loses 2 seconds per day, find approximately what alteration should be made in its length.

23. A seconds pendulum at sea-level gains n secs. per day: to what height must it be elevated in order to keep true time?

24. A seconds pendulum is lengthened by heat by the 100th part of itself; how many seconds does it lose in a day?

25. If a pendulum make 40,000 vibrations in 6 hours at the level of the sea, how many vibrations will it make in the same time at a height of 10,560 ft. above the sea-level?

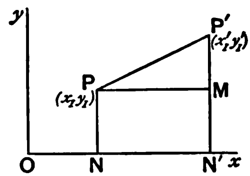
26. Shew that the number of vibrations made in a day by a pendulum of given length varies, with a variation in the latitude, as the square root of the length of the seconds pendulum at the place of observation.

CHAPTER XIX.

MOTION OF CENTRE OF GRAVITY.

171. *To find the velocity of the centre of gravity of any number of heavy particles of given masses moving uniformly in a plane.*

Let the position and motion of the balls be referred to two rectangular axes Ox, Oy in the plane of motion.



Let m_1, m_2, ... be the masses of the particles; (x_1, y_1), (x_2, y_2), ... the co-ordinates of their positions at the instant under consideration; $(\bar{x}, \bar{y})$ the co-ordinates of their C. G. at the instant under consideration; (x_1', y_1'), (x_2', y_2'), ... and $(\bar{x}', \bar{y}')$ the corresponding co-ordinates after an interval t_1; u_1, u_2, ... the velocities of the particles parallel to the axis of x; v_1, v_2, ... the velocities of the particles parallel to the axis of y; $\bar{u}$, $\bar{v}$ the velocities of their C. G. parallel to the axes.

Then from the figure we see that

$$x_1' - x_1 = PM = u_1 t. \qquad (s = ut)$$

Similarly $$x_2' - x_2 = u_2 t,$$

....

$$x_n' - x_n = u_n t.$$

Also from Statics we know that

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n},$$

and $$\bar{x}' = \frac{m_1x_1' + m_2x_2' + \ldots m_nx_n'}{m_1 + m_2 + \ldots + m_n},$$

$$\therefore \bar{x}' - \bar{x} = \frac{m_1(x_1' - x_1) + m_2(x_2' - x_2) + \ldots + m_n(x_n' - x_n)}{m_1 + m_2 + \ldots + m_n};$$

i.e. $$\bar{u}t = \frac{m_1u_1t + m_2u_2t + \ldots + m_nu_nt}{m_1 + m_2 + \ldots + m_n},$$

$$\therefore \bar{u} = \frac{m_1u_1 + m_2u_2 + \ldots + m_nu_n}{m_1 + m_2 + \ldots + m_n} = \frac{\Sigma(mu)}{\Sigma(m)} \text{ suppose.}$$

Similarly

$$\bar{v} = \frac{m_1v_1 + m_2v_2 + \ldots + m_nv_n}{m_1 + m_2 + \ldots + m_n} = \frac{\Sigma(mv)}{\Sigma(m)}.$$

Whence $\bar{u}$ and $\bar{v}$ being known, the resultant velocity of the C. G. of the system is known, viz. $\sqrt{\bar{u}^2 + \bar{v}^2}$.

Also if this resultant makes angle θ with the axis of x,

$$\tan\theta = \frac{\bar{v}}{\bar{u}} = \frac{\Sigma(mv)}{\Sigma(mu)}.$$

Thus we see that the velocity of the C. G. of a system of co-planar particles in any given direction in their plane is equal to the sum of the momenta of the particles in that direction divided by the sum of the masses.

172. *To find the acceleration of the centre of gravity of a system of heavy particles moving in a plane with given uniform accelerations.*

Let $m_1, m_2 \ldots m_n$ be the masses of the particles.

Take two rectangular axes Ox, Oy in the plane of motion.

Let u_1, v_1 be the velocities of m_1 parallel to the axes at the instant under consideration;

α_1, β_1 the accelerations of m_1 parallel to the axes;

$\bar{u}$, $\bar{v}$ the velocities of their C. G. parallel to the axes at the instant under consideration, and u_1', v_1', $\bar{u}'$, $\bar{v}'$ the corresponding velocities after an interval t.

Also let $\bar{\alpha}$, $\bar{\beta}$ be the accelerations of their C. G. parallel to the axes.

Then we have

$$u_1' = u_1 + \alpha_1 t. \qquad (v = u + ft)$$

Similarly

$$u_2' = u_2 + \alpha_2 t,$$

$$\dots\dots\dots\dots\dots$$

$$u_n' = u_n + \alpha_n t.$$

Also, by the preceding article,

$$\bar{u} = \frac{m_1 u_1 + m_2 u_2 + \dots + m_n u_n}{m_1 + m_2 + \dots + m_n}$$

and

$$\bar{u}' = \frac{m_1 u_1' + m_2 u_2' + \dots + m_n u_n'}{m_1 + m_2 + \dots + m_n},$$

$$\therefore\ \bar{u}' - \bar{u} = \frac{m_1 (u_1' - u_1) + m_2 (u_2' - u_2) + \dots + m_n (u_n' - u_n)}{m_1 + m_2 + \dots + m_n},$$

i.e.

$$\bar{\alpha} t = \frac{m_1 \alpha_1 t + m_2 \alpha_2 t + \dots + m_n \alpha_n t}{m_1 + m_2 + \dots + m_n},$$

$$\therefore\ \bar{\alpha} = \frac{m_1 \alpha_1 + m_2 \alpha_2 + \dots + m_n \alpha_n}{m_1 + m_2 + \dots + m_n} = \frac{\Sigma (m\alpha)}{\Sigma (m)}.$$

Similarly

$$\bar{\beta} = \frac{m_1 \beta_1 + m_2 \beta_2 + \dots + m_n \beta_n}{m_1 + m_2 + \dots + m_n} = \frac{\Sigma (m\beta)}{\Sigma (m)}.$$

Whence $\bar{\alpha}$ and $\bar{\beta}$ being known, the resultant acceleration, viz. $\sqrt{\bar{\alpha}^2 + \bar{\beta}^2}$ is also known; and if it makes an angle θ with the axis of x

$$\tan \theta = \frac{\bar{\beta}}{\bar{\alpha}} = \frac{\Sigma (m\beta)}{\Sigma (m\alpha)}.$$

173. *When two smooth balls impinge upon one another the motion of their centre of gravity is unaltered by the impact.*

Let m_1, m_2 be the masses of the balls.

First, when the impact is direct.

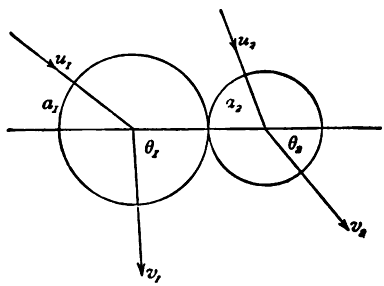
Let u_1, u_2 be their velocities before impact, v_1, v_2 their velocities after impact, $\bar{u}$, $\bar{v}$ the velocities of their centre of gravity before and after impact respectively.

No momentum is lost by the impact,

$$\therefore\ m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2.$$

Hence
$$\bar{v} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$
$$= \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$
$= \bar{u}$, which proves the proposition.

Secondly, when the impact is oblique.



Let u_1, u_2 be the velocities of the balls before impact, making angles α_1, α_2 with the line of impact; v_1, v_2 their velocities after impact, making angles θ_1, θ_2 with the line of impact.

Let $\bar{u}$ be the velocity of their C. G. before impact, making angle α with the line of impact.

Let $\bar{v}$ be the velocity of their C. G. after impact, making angle θ with the line of impact.

$$\bar{v}\sin\theta = \frac{m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2}{m_1 + m_2}$$

$$= \frac{m_1 u_1 \sin\alpha_1 + m_2 u_2 \sin\alpha_2}{m_1 + m_2}$$

(since the motion at right angles to the line of impact is unaffected by the impact)

$$= \bar{u}\sin\alpha.$$

Also $$\bar{v}\cos\theta = \frac{m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2}{m_1 + m_2}$$

$$= \frac{m_1 u_1 \cos\alpha_1 + m_2 u_2 \cos\alpha_2}{m_1 + m_2}$$

(since no momentum is lost in the line of impact)

$$= \bar{u}\cos\alpha,$$

$$\therefore\ \bar{v}\sin\theta = \bar{u}\sin\alpha$$

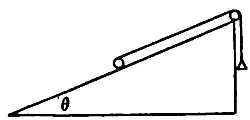
and $$\bar{v}\cos\theta = \bar{u}\cos\alpha;$$

whence $$\theta = \alpha$$

and $\bar{v} = \bar{u}$, which proves the proposition.

174. *Example. A ball, hanging freely, draws an equal ball up an inclined plane of elevation θ, by means of a light string passing over the top of the plane. Find the acceleration of their centre of gravity.*

Let f be the acceleration of each ball, a the horizontal acceleration of their C. G., β the vertical acceleration of their C. G., *downwards*.



Then

$$f = \frac{(m - m\sin\theta)g}{2m} \text{ (Art. 61)}$$

$$= \frac{1-\sin\theta}{2}\cdot g \quad \text{.......................................(i)},$$

where m is the mass of each ball.

Also $$\alpha = \frac{mf\cos\theta}{2m} = \frac{f\cos\theta}{2} \qquad \text{[Art. 172]}$$

(the hanging body has no horizontal acceleration), and

$$\beta = \frac{mf - mf\sin\theta}{2m} = \frac{f(1-\sin\theta)}{2}, \qquad \text{[Art. 172]}$$

therefore their resultant acceleration

$$= \sqrt{\frac{f^2\cos^2\theta + f^2(1-\sin\theta)^2}{4}}$$

$$= \frac{f}{2}\sqrt{\cos^2\theta + (1-\sin\theta)^2} = \frac{f}{2}\sqrt{2(1-\sin\theta)}$$

$$= \frac{g(1-\sin\theta)}{4}\sqrt{2(1-\sin\theta)} = \frac{g\sqrt{2}}{4}(1-\sin\theta)^{\frac{3}{2}} \qquad \text{from (1)}$$

$$\frac{g\sqrt{2}}{4}\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^3.$$

Also the angle it makes with the horizon

$$= \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}\right] = \tan^{-1}\left[\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}\right] = \frac{\pi}{4} - \frac{\theta}{2}.$$

EXAMPLES. XIX.

1. Two masses m_1 and m_2 are connected by a light string over a smooth fixed pulley; find the acceleration of their centre of gravity.

2. Two masses move with uniform velocities along two straight lines inclined at a given angle; shew that their centre of gravity describes a straight line with uniform velocity.

3. Find the velocity of the centre of gravity of two masses of 4 lbs. and 5 lbs., moving in parallel lines with velocities of 9 and 18 ft. per sec. respectively,

(1) when they move in the same direction;

(2) when they move in opposite directions.

4. Two perfectly elastic balls are dropped from two points not in the same vertical line, and strike against a perfectly elastic horizontal plane: shew that their centre of gravity will never re-ascend to its original height, unless the initial heights of the balls be in the ratio of two square numbers.

5. A mass m_1, lying on a smooth horizontal table, is connected by a light inextensible string with another mass m_2 hanging freely over the edge of the table. Determine the motion of their centre of mass.

6. Three equal balls are projected at the same instant from the same point with the same velocity, 1600 ft. per sec., and at elevations 30°, 45°, 60° respectively; find the height of the centre of gravity of the balls above the horizontal plane through the point of projection after 12 seconds.

7. Three equal particles are placed at the angles of the triangle ABC, and move respectively from A to B, B to C, and C to A, the three motions being accomplished simultaneously with uniform velocity and in the same time; prove that their centre of gravity remains at rest.

8. Two masses, $3m$ and m, move on two planes each inclined to the horizon at an angle of 45°, and are connected by a fine string which passes over their common vertex. Find the acceleration of their centre of gravity. What is its locus?

9. A body weighing 2 lbs. is projected with a velocity of 20 ft. per second at an angle of 60° to the horizon; another weighing 3 lbs. is at the same time projected from the same point at an angle of 30° with a velocity of 40 ft. per second. Find to two places of decimals the height to which their centre of gravity mounts, and the distance at which it meets the horizontal plane through the point of projection.

10. A weight P hanging vertically just supports a weight W in that system of pulleys in which there is only one string. Shew that, if the masses of the pulleys be neglected, and if P and W be interchanged, then their common centre of gravity will descend with an acceleration equal to

$$\frac{(W-P)^2 g}{W^2 - WP + P^2}.$$

11. Two equal particles start simultaneously from A, one sliding down a plane AB, and the other falling freely down the height AC; prove that if CD be drawn perpendicular to AB, their centre of gravity describes the line from A to the middle point of CD with uniform acceleration.

12. Two bodies are projected together from a point with different velocities and elevations. Shew that their centre of gravity moves as if it were a heavy particle projected from the same point.

13. Two weights are connected by a string passing over the common vertex of two smooth inclined planes on which the weights rest and balance each other; shew that if the weights are set in motion (the string remaining stretched) the centre of gravity of the two weights will move in a horizontal straight line.

14. Two equal particles start simultaneously from the origin and describe the straight lines $x+y=0$, $x-7y=0$ with uniform velocities $9u$, $10u$ respectively, the motion of each particle being to the right of the axis of y. Shew that the centre of gravity of the particles moves with uniform velocity $\frac{17u}{2}$, and find the equation of its path.

CHAPTER XX.

UNITS.

175. The three *fundamental* units are the units of *length, time,* and *mass*; and all physical quantities can be expressed in terms of these three. Hence all units, other than the fundamental, are called *derived* units.

Thus the ordinary unit of velocity, i.e. a velocity of one foot per second, is a *derived* unit.

The measure of any quantity multiplied by the unit employed is always the same.

When we speak of 4 yards, we mean that 4 is the measure when a yard is the unit. Also 4 yards $=12$ feet, hence 12 is the measure of the same length when one foot is the unit. Thus

$$4 \text{ yards} = 12 \text{ feet} = 144 \text{ inches}.$$

In the same way if a, b, c be the measures of a quantity when $[A]$, $[B]$, $[C]$ are the units respectively,

$$a\,[A] = b\,[B] = c\,[C].$$

The same idea is often thus expressed:—

The measure of a quantity varies inversely as the unit employed.

176. Unit of area. Let a, l, b denote the measures of the area, length, and breadth respectively of a rectangle when $[A]$ and $[L]$ are respectively the units of area and length.

Now $$a = l \times b.$$

But $$a \propto \frac{1}{[A]}, \quad l \propto \frac{1}{[L]}, \quad \text{and } b \propto \frac{1}{[L]};$$

since the measure of a quantity varies inversely as the unit employed.

$$\therefore \frac{1}{[A]} \propto \frac{1}{[L]^2},$$

i.e. $$[A] \propto [L]^2.$$

Hence the unit of area varies as the square of the unit of length.

Unit of volume. Similarly if $[C]$ denote the unit of volume, and d the depth of a rectangular parallelepiped;

$$c = l \times b \times d.$$

But $$c \propto \frac{1}{[C]}, \quad l \propto \frac{1}{[L]}, \quad b \propto \frac{1}{[L]}, \quad d \propto \frac{1}{[L]}, \quad \text{(Art. 175.)}$$

$$\therefore \frac{1}{[C]} \propto \frac{1}{[L]^3},$$

i.e. $$[C] \propto [L]^3.$$

Hence the unit of volume varies as the cube of the unit of length.

177. **Unit of velocity.** Let s be the measure of the space described in time t by a body moving with uniform velocity v, when $[L]$, $[T]$, $[V]$ denote the units of length, time, and velocity respectively.

Then $$s = vt.$$

But since the measure of a quantity varies inversely as the unit employed,

$$s \propto \frac{1}{[L]}, \quad v \propto \frac{1}{[V]}, \quad t \propto \frac{1}{[T]},$$

$$\therefore \frac{1}{[L]} \propto \frac{1}{[V]} \times \frac{1}{[T]},$$

i.e.
$$[L] \propto [V]\,[T],$$
$$\therefore\ [V] \propto [L]\,[T]^{-1}.$$

Hence the unit of velocity varies directly as the unit of length and inversely as the unit of time.

Also the unit of velocity is of one dimension in length, and minus one dimension in time.

178. Unit of acceleration. Let a body, moving with acceleration f, describe a space s from rest in time t, when $[F]$ $[L]$ $[T]$ are respectively the units of acceleration, length, and time.

Then
$$s = \tfrac{1}{2} f t^2.$$

But since the measure of a quantity varies inversely as the unit employed,

$$s \propto \frac{1}{[L]},\quad f \propto \frac{1}{[F]},\quad t \propto \frac{1}{[T]},$$
$$\therefore\ \frac{1}{[L]} \propto \frac{1}{[F]} \times \frac{1}{[T]^2},$$
i.e.
$$[L] \propto [F]\,[T]^2,$$
or
$$[F] \propto [L]\,[T]^{-2}.$$

Hence the unit of acceleration varies directly as the unit of length, and inversely as the square of the unit of time.

Also we see that the unit of acceleration is of one dimension in length and minus two dimensions in time.

179. Unit of force. With the same notation as that employed in the previous articles, if p be the measure of a force when $[P]$ is the unit of force,

$$p = mf.$$

But the measure of a quantity varies inversely as the unit employed,

$$\therefore\ p \propto \frac{1}{[P]},\quad m \propto \frac{1}{[M]},\quad f \propto \frac{1}{[F]},$$

therefore, as before,
$$[P] \propto [M][F]$$
$$\propto [M][L][T]^{-2}. \quad \text{(Art. 178.)}$$

Hence the unit of force varies directly as the units of mass and length, and inversely as the square of the unit of time.

Also the unit of force is of one dimension in mass, one dimension in length, and minus two dimensions in time.

180. Unit of momentum. Momentum is measured by the product mv, hence if $[K]$ denote the unit of momentum, as in the previous articles,
$$[K] \propto [M][V]$$
$$\propto [M][L][T]^{-1}. \quad \text{(Art. 177.)}$$

Hence the unit of momentum varies directly as the units of mass and length, and inversely as the unit of time.

Unit of impulse. Impulse is measured by change of momentum, and therefore the unit of impulse must involve the same fundamental units as that of momentum. Therefore the unit of impulse
$$\propto [M][L][T]^{-1}.$$

Thus both the units of momentum and impulse are of one dimension in mass, one dimension in length, and minus one dimension in time.

181. Unit of work. If w be the measure of the work done by force p in moving its point of application through space s, where $[W]$ is the unit of work,
$$w = ps.$$

But the measure of any quantity varies inversely as the unit employed,
$$\therefore\ w \propto \frac{1}{[W]}, \quad p \propto \frac{1}{[P]}, \text{ and } s \propto \frac{1}{[L]},$$
$$\therefore\ \frac{1}{[W]} \propto \frac{1}{[P]} \times \frac{1}{[L]},$$

$$[W] \propto [P][L]$$
$$\propto [M][L]^2[T]^{-2}. \qquad \text{(Art. 179.)}$$

Therefore the unit of work varies directly as the unit of mass, and the square of the unit of length, and inversely as the square of the unit of time.

Also the unit of work is of one dimension in mass, two dimensions in length, and minus two dimensions in time.

182 **Unit of kinetic energy.** Let E be the measure of the kinetic energy of a body of mass m, moving with velocity v; then with the usual notation, since

$$E = \tfrac{1}{2} mv^2,$$

we have as in the previous articles

$$[E] \propto [M][V]^2$$
$$\propto [M][L]^2[T]^{-2}. \qquad \text{(Art. 177.)}$$

Thus the unit of kinetic energy is the same as that of work.

183. **Unit of rate of work or power.**

Let h be the power at which work w is done in time t, then

$$ht = w,$$

and therefore as in preceding articles

$$[H][T] \propto [W],$$

i.e.
$$[H] \propto [W][T]^{-1}$$
$$\propto [M][L]^2[T]^{-3}. \qquad \text{(Art. 181.)}$$

Unit of angular velocity.

With the usual notation

$$v = r\omega,$$

and therefore, as before,

$$[V] \propto [L][\Omega],$$

i.e.

$$[\Omega] \propto [V][L]^{-1}$$

$$\propto [T]^{-1}. \qquad \text{(Art. 177.)}$$

184. Ex. i. *If 240 lbs. be the unit of mass, one mile the unit of length, one minute the unit of time; find the unit of force.*

Let $[P']$ be the required unit of force.

With the usual notation

$$\frac{[P']}{[P]} = \frac{[M'][L'][T']^{-2}}{[M][L][T]^{-2}} \quad \text{..................(1).}$$

Now let $[P]$, $[M]$, $[L]$, $[T]$ denote the ordinary ft.-sec.-poundal units, and express all the units in the above equation in ft.-secs., lbs., poundals respectively.

We then have

$$[P'] = (240)(5280)(60)^{-2} \text{ poundals}$$

$$= 352 \text{ poundals}$$

$$= 11 \text{ lbs. wt.}$$

N.B. In examples like the above we must remember that the ratios must all be ratios of like quantities; thus we have taken $[P']$ and $[P]$ both in poundals, $[M']$ and $[M]$ both in pounds, and so on.

Also since $[P]$, $[M]$, $[L]$, $[T]$ stand for the ordinary British units, when $[M]$ is unity viz. 1 lb. and $[L]$ is unity viz. 1 foot, and $[T]$ is unity viz. one second, then $[P]$ is also unity viz. 1 poundal.

Ex. ii. *Find the measure of an acceleration of f ft.-sec. units when m yards and n minutes are the units of length and time.*

Let f' be the measure required; then with the usual notation

$$\frac{f'}{f} = \frac{[F]}{[F']} = \frac{[L][T]^{-2}}{[L'][T']^{-2}}$$

$$= \left(\frac{1}{3m}\right)\left(\frac{1}{60n}\right)^{-2} \text{ ....(i)}$$

$$= \frac{3600n^2}{3m} = \frac{1200n^2}{m},$$

$$\therefore f' = \frac{1200\, n^2 f}{m}.$$

N.B. In (i) we must express $[L]$ and $[L']$ in terms of the same unit, viz. one foot; also $[T]$ and $[T']$ must be expressed in terms of the same unit, viz. one second.

Ex. iii. *If the acceleration of a falling body be the unit of acceleration, and the velocity acquired by it in 5 secs. be the unit of velocity; find the units of length and time.*

A body moving with acceleration g acquires a velocity

$5g$ ft. per sec. in 5 secs., $\quad (v = u + ft)$

therefore $5g = 160$ ft. per sec. is the unit of velocity.

Let $[L']$, $[F']$, $[T']$, $[V']$ denote the new units of length, force, time, and velocity expressed in ft.-poundals, seconds, and ft. per sec. respectively, and $[L]$, $[F]$, $[T]$, $[V]$ the ordinary ft.-sec.-poundal units.

Then $$\frac{[L']}{[L]} = \frac{[F'][T']^2}{[F][T]^2};$$

therefore in this case

$$[L'] = (32)[T']^2 \text{ ....(1).}$$

Also $$\frac{[L']}{[L]} = \frac{[V'][T']}{[V][T]};$$

therefore here $$[L'] = 160[T'] \text{ ....(2).}$$

Therefore dividing (1) and (2)

$$1 = \frac{[T']}{5},$$

$$[T'] = 5 \text{ seconds}$$

therefore from (1) $\quad [L'] = 32\,(5)^2 = 800$ feet.

Ex. iv. *The acceleration due to gravity is represented by* 80, *the unit of time is* 5 *secs.; find the unit of length.*

With the same notation, and expressing all units in ft.-secs., lbs., poundals as before,

$$\frac{[L']}{[L]} = \frac{[F']\,[T']^2}{[F]\,[T]^2},$$

$$\therefore\ [L'] = [F']\,25.$$

Also

$$80\,[F'] = 32\ [\text{ft.-sec. units}],$$

$$\therefore\ 80\,\frac{[L']}{25} = 32,$$

i.e. $\quad [L'] = \dfrac{32 \times 25}{80} = 10$ feet.

EXAMPLES. XX.

1. If the unit of time be half a minute, and the unit of length half a mile, find the unit of velocity.

2. If the unit of velocity be a velocity of 40 miles an hour, and the unit of time be 5 minutes, find the unit of length.

3. Find the unit of length when the acceleration of gravity is represented by the number 16, and the unit of time is 3 seconds.

4. A particle describes 2 miles uniformly in 11 minutes; if 4 be the measure of its velocity, and 8 ft. the unit of length, find the unit of time.

5. If the unit of time be one minute, and the unit of velocity a velocity of 4 miles per hour, find the unit of length.

6. If 4 yds. be the unit of length, and 4 secs. the unit of time, find the unit of acceleration.

7. If f denote an acceleration when ft. and secs. are the units, what will be its measure when m ft. and n secs. are the units?

8. The acceleration of gravity is represented by 4, the unit of time is 2 secs.; what is the unit of length?

9. The acceleration of gravity is represented by 3200, 4 ft. is the unit of length; find the unit of time.

10. The measure of a force is 12 when ft., secs., pounds are the units of length, time, and mass respectively; find its measure when miles, minutes and ounces are the units.

11. If an acceleration be represented by 10, when feet and seconds are taken as units, what must be the unit of time in order that the same acceleration may be represented by $67\frac{1}{2}$ when a yard is the unit of length?

12. A body, moving uniformly, passes over a mile in 15 minutes; if 64 be the measure of the velocity, and 2 minutes the unit of time, find the unit of length.

13. What is the measure of the acceleration due to gravity when a foot and half a second are units of length and time?

14. A velocity of one foot per second is changed uniformly in one minute to a velocity of one mile an hour. Express numerically the rate of change when a yard and a minute are the units of space and time.

15. If the weight of 4 lbs. be the unit of force, an acceleration of 16 ft.-sec. units the unit of acceleration, find the unit of mass.

16. The unit of force is the weight of 1 lb., the unit of mass is the mass of 4 lbs., and the unit of velocity a velocity of 8 ft. per sec.; find the units of time and length.

17. If the acceleration due to gravity be taken as the unit of acceleration, and the velocity generated in half a minute be the unit of velocity, find the unit of length.

18. What is the measure of the acceleration due to gravity when a mile and 11 seconds are the units of length and time?

19. If the unit of energy be 1000 ft.-pounds, and 20 lbs. the unit of mass, find the unit of velocity.

20. In a certain system of absolute units the acceleration of gravity is denoted by 4, the momentum of a 100 lbs. shot moving at 1200 ft. per sec. by 2, and its kinetic energy by 5; find the units of time, length and mass.

21. If f_1, f_2 be the measures of an acceleration when $m+n$ secs. and $m-n$ secs. are the respective units of time, and a ft. and b ft. the respective units of length, shew that the measure becomes $\frac{1}{c}(\sqrt{f_1 a}+\sqrt{f_2 b})^2$ when $2m$ secs. is the unit of time, and c ft. the unit of length.

22. If the acceleration of gravity be denoted by 2400, a yard being the unit of length, find the unit of time.

23. The measures of an acceleration and a velocity when referred to $(a+b)$ ft., $(m+n)$ secs., and $(a-b)$ ft., $(m-n)$ secs. respectively are in the inverse ratio of their measures when referred to $(a-b)$ ft., $(m-n)$ secs., and $(a+b)$ ft., $(m+n)$ secs.; their measures when referred to a ft., m secs. and b ft., n secs. are as $ma : nb$; shew that $\frac{n^2}{m^2}=1-\frac{b^4}{a^4}$.

24. If the unit of time be 4 minutes and the unit of length 3 yards, find the measure of g.

25. If a second be the unit of time, the acceleration due to gravity (981 in C.G.S. units) the unit of acceleration, and a kilogramme the unit of mass, find the unit of energy in ergs.

CHAPTER XXI.

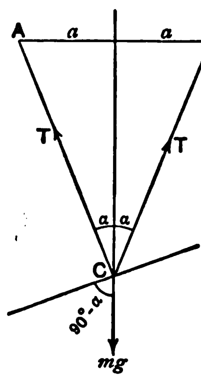
INITIAL ACTIONS, TENSION, MOTION, ETC.

185. We shall here consider such problems as the following:—

Suppose a system of particles at rest, or in motion, but under certain constraining forces. Let one of the constraining forces be suddenly removed; it is then required to find the instantaneous change produced, either, in any of the other constraining forces, or in the motion of any of the particles.

186. *A particle of mass m is attached by two separate inelastic strings of equal length to two points in the same horizontal line. Find the instantaneous change in the tension of one string when the other is suddenly cut. Find also the initial acceleration of the particle when the string is cut.*

Let A, B be the two fixed points to which the strings are attached and let $AB = 2a$. Let the particle hang in equilibrium at the point C, and let l be the length of each string, so that $AC = BC = l$; also let $\angle ACB = 2\alpha$. Since there is equilibrium we have, resolving vertically,



$$2T\cos\alpha = mg,$$

$$\therefore\ T = \frac{mg}{2\cos\alpha}.$$

Let T' be the tension of the string AC, immediately after BC is cut.

The string AC remains constant in

length, therefore the particle begins to move at right angles to AC; i.e. there is no motion in the direction AC.

Therefore resolving in that direction,

$$T' = mg \cos \alpha.$$

Hence the instantaneous change of tension required

$$= T' - T = mg \cos \alpha - \frac{mg}{2 \cos \alpha}$$

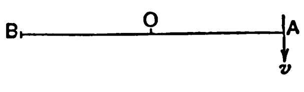
$$= \frac{mg}{2 \cos \alpha} (2 \cos^2 \alpha - 1)$$

$$= \frac{mg \cos 2\alpha}{2 \cos \alpha} \text{ where } \cos \alpha = \frac{\sqrt{l^2 - a^2}}{l}.$$

Thus we see that if $\alpha > 45°$, the tension is suddenly diminished by cutting BC ($\cos 2\alpha$ being then negative); whereas if $\alpha < 45°$ the tension is suddenly increased.

Again the resultant force in the direction of motion, i.e. at right angles to AC, immediately after the string is cut is $mg \sin \alpha$; therefore the initial acceleration of the particle is then $g \sin \alpha$ at right angles to AC.

187. *Two equal particles connected by a string are revolving in a circle on a smooth horizontal table with equal speeds, the string being always a diameter of the circle. If one of the particles be suddenly pinned to the table, find the instantaneous change in the tension of the string.*



Let m be the mass of each particle, r the radius of the circle each describes; v their velocity, T_1 the tension of the string when they both revolve, T_2 the tension immediately after one of them is suddenly pinned to the table. Let the one particle be at A when the other is stopped at B, then the velocity of the particle at A, being *at that instant* in a direction at right angles to AB, will be the same *immediately after* B is stopped as before, viz. v.

Before the particle at B was pinned, the particle at A was describing a circle of radius r,

$$\therefore\ T_1 = \frac{mv^2}{r}.$$

Immediately after the particle at B is pinned, the other particle proceeds to describe a circle of radius $2r$,

$$\therefore\ T_2 = \frac{mv^2}{2r}.$$

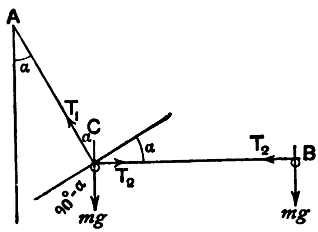
Therefore the instantaneous change in the tension

$$= T_1 - T_2 = \frac{mv^2}{r} - \frac{mv^2}{2r}$$

$$= \frac{mv^2}{2r}.$$

188. *A string AB has one end A attached to a fixed point, a mass m attached to the other end B, and an equal mass attached to its middle point C. The string is held so that AC makes an angle α with the vertical and BC is horizontal, and the mass at B is then suddenly released; find the instantaneous changes in the tensions of the two portions of the string.*

Let P_1 be the tension of AC, P_2 that of BC before the particle at B is released. Consider the equilibrium of the point C.



Resolving vertically

$$P_1 \cos\alpha = mg,\ \therefore P_1 = \frac{mg}{\cos\alpha} \quad \text{(i)}.$$

Resolving at right angles to AC

$$P_2 \cos\alpha = mg\sin\alpha,\ \therefore\ P_2 = mg\tan\alpha \ldots\ldots\ldots \text{(ii)}.$$

Again, let T_1 and T_2 be the tensions of AC and BC respectively *immediately after* the particle at B is released. Consider the particle at C.

AC is constant in length, therefore ***at the instant*** the resultant force on the particle in the direction AC is zero. Therefore resolving in that direction,

$$T_1 - T_2 \sin \alpha = mg \cos \alpha \ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Also the resultant force perpendicular to AC

$$= mg \sin \alpha - T_2 \cos \alpha,$$

therefore the acceleration of the particle at C perpendicular to AC

$$= \frac{mg \sin \alpha - T_2 \cos \alpha}{m},$$

and the component of this acceleration in the direction BC

$$= \left(\frac{mg \sin \alpha - T_2 \cos \alpha}{m}\right) \cos \alpha \ldots\ldots\ldots\ldots\text{(iv)}.$$

Next consider the particle at B.

At the beginning of its motion, the resultant force in direction $BC = T_2$, therefore the acceleration in that direction

$$= \frac{T_2}{m}.$$

But BC is of invariable length,

$$\therefore \frac{T_2}{m} = \left(\frac{mg \sin \alpha - T_2 \cos \alpha}{m}\right) \cos \alpha \text{ from (iv)},$$

whence

$$T_2 = \frac{mg \sin \alpha \cos \alpha}{1 + \cos^2 \alpha},$$

and from (iii),

$$T_1 = T_2 \sin \alpha + mg \cos \alpha$$

$$= \frac{mg \cos \alpha}{1 + \cos^2 \alpha} (\sin^2 \alpha + 1 + \cos^2 \alpha)$$

$$= \frac{2mg \cos \alpha}{1 + \cos^2 \alpha};$$

therefore the instantaneous change in the tension of AC

$$= P_1 - T_1 = \frac{mg}{\cos\alpha} - \frac{2\,mg\cos\alpha}{1+\cos^2\alpha} \text{ from (i)}$$

$$= \frac{mg\sin^2\alpha}{\cos\alpha\,(1+\cos^2\alpha)},$$

and the instantaneous change in the tension of BC

$$= P_2 - T_2 = \frac{mg\sin\alpha}{\cos\alpha} - \frac{mg\sin\alpha\cos\alpha}{1+\cos^2\alpha} \text{ from (ii)}$$

$$= \frac{mg\sin\alpha}{\cos\alpha\,(1+\cos^2\alpha)}.$$

EXAMPLES. XXI.

The harder examples are marked with an asterisk *.

1. A mass of 30 ozs. is supported by two strings, each 5 ft. long, attached to two points 6 ft. apart in a horizontal line. One string is cut: find the instantaneous change in the tension of the other string.

2. A heavy uniform string rests on a smooth horizontal table with one end fastened to the table and one half its length hanging over the edge of the table; prove that when the string is suddenly set free the vertical pressure on the table will be instantaneously diminished by one quarter of the weight of the string.

*3. Three equal smooth spheres are placed in contact on a smooth horizontal plane, and are connected where they touch. A fourth equal sphere is placed so as to be supported by the other three. If the connections between the lower spheres be simultaneously broken, find the instantaneous change of pressure between the upper ball and each of the lower ones.

4. A particle of mass 5 lbs. is attached to a string passing over a smooth pulley and fastened at the other end to a body of mass 4 lbs. lying on a table, the line joining the particle on the table to the pulley making an angle of 30° with the horizon. If the string be at first slack, and become stretched after the 5 lbs. mass has fallen through one foot, find the impulsive tension of the string, and the acceleration of the falling particle immediately after the string is stretched.

5. Three particles of equal masses are attached at equal intervals to a string of negligible mass. The middle particle is held at a point

A of a smooth horizontal plane, and the other two describe the same circle about it in the same sense with the same uniform speed, so that the line joining them subtends an angle a at the point A. Prove that if the middle particle be let go, the tension of either part of the string is suddenly diminished in the ratio $1 : 2+\cos a$.

*6. A string ABC has one end A fastened to a fixed point, and equal masses (m) attached at B and C. It is held so that AB is inclined at 60° to the horizon, and BC at an angle 30° below the horizon: find the instantaneous change in the tensions of the two portions of the string when the particle at C is suddenly released.

7. Two unequal particles are connected by a light inextensible string, which is stretched taut. The particles are moving with the same velocity v perpendicular to the string when it strikes a small smooth fixed obstacle. Prove that immediately after the impact the tension in the string is $\frac{mv^2}{a}$: where m, a are the harmonic means between the masses of the particles and between the distances of the particles from the obstacle.

*8. A light string fixed at one end has two particles attached to it, m_1 at its middle point, and m_2 at its free end; m_2 is held in the same horizontal line as the fixed end, the two segments of the string making an angle a with the horizon. Prove that, if m_2 is let go, the initial tension of the string at the fixed end is

$$\frac{m_1(m_1+m_2)g\sin a}{m_1+m_2\sin^2 2a}, \text{ provided } a<\frac{\pi}{4}.$$

What happens if $a>\frac{\pi}{4}$?

9. A particle is whirled round in a vertical plane, being attached to a fixed point by a fine string. If when the particle has just speed enough to carry it round in a complete circle its speed at the lowest point is doubled, find how the tension of the string in that position is altered.

*10. Two particles of masses m_1 and m_2 are attached by a fine inextensible string and move in a smooth horizontal plane with the same velocity v perpendicularly to the string. The string suddenly strikes against a smooth fixed post of small radius at distances l_1 and l_2 from m_1 and m_2. Prove that the initial acceleration of the point of the string initially touching the post is

$$\frac{v^2\left(\frac{m_1}{l_1}\sim\frac{m_2}{l_2}\right)}{m_1+m_2}.$$

11. Two particles of masses m, $2m$ respectively, are fastened to the ends of a string of length $2a$, and move at right angles to the string with velocity v on a smooth horizontal table, the string being straight. If the string at its middle point strikes a fixed vertical peg on the table, find its tension immediately after the impact.

12. A loaded cannon is suspended from a fixed horizontal beam and rests with its axis horizontal and perpendicular to the beam, the supporting ropes being equally inclined to the vertical. If v be the initial velocity of the ball, whose mass is $\frac{1}{n}$th that of the cannon, and h the depth of the cannon below the beam, show that when it is fired off the tension of each rope will be changed in the ratio

$$v^2+n^2gh : n(n+1)gh.$$

*13. Two equal particles, mass m, are connected by equal taut strings, length l, to another particle, mass M. This particle is struck by a blow I in a line bisecting the angle $2a$ between the strings. Show that the initial angular velocity of either string is

$$\frac{I \sin a}{l(M+2m\cos^2 a)}.$$

CHAPTER XXII.

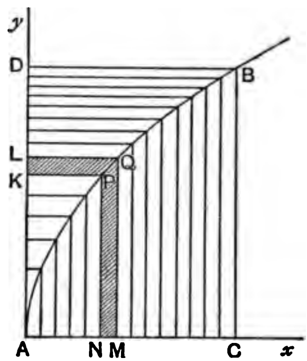
MOTION OF A PARTICLE MOVING UNDER A VARIABLE ACCELERATION.

189. *To find the area of a parabola.*

Let AB be a parabola whose equation is $y^2 = 4ax$.

It is required to find the area of the portion ACB.

Divide the abscissa AC into a number of small parts and draw ordinates to meet the curve in P, Q .... Also draw parallels to AC to meet the axis of y in K, L, etc., thus dividing the areas ACB, ADB into a number of small strips.



Let (x, y) be the co-ordinates of P, $(x+\Delta x, y+\Delta y)$ of Q, *near to* P.

$$\frac{\text{Area } PM}{\text{Area } PL} = \frac{PN \, . \, NM}{PK \, . \, KL} = \frac{y\Delta x}{x\Delta y}.$$

Now $$y^2 = 4ax$$

and $$(y + \Delta y)^2 = 4a(x + \Delta x);$$

$\therefore$ subtracting, $2y\Delta y + \Delta y^2 = 4a\Delta x$,

$$2y + \Delta y = 4a\frac{\Delta x}{\Delta y}.$$

$\therefore$ when the strip PM is *very narrow* $2y = 4a\dfrac{\Delta x}{\Delta y}$,

$$\text{or } \frac{\Delta x}{\Delta y} = \frac{y}{2a}.$$

$$\therefore \frac{\text{area } PM}{\text{area } PL} = \frac{y^2}{2ax} = \frac{4ax}{2ax} = 2.$$

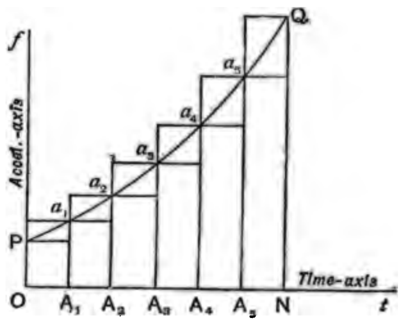
$\therefore$ adding up all the strips in each of the areas ACB, ADB,

$$\frac{\text{area } ACB}{\text{area } ADB} = \frac{2}{1}.$$

$$\therefore \text{ Componendo } \frac{\text{area } ACB}{\text{rect. } ACBD} = \frac{2}{3},$$

$$\text{or area } ACB = \tfrac{2}{3}\,.\,AC\,.\,CB.$$

190. *Given the acceleration-time curve for the motion of a particle, to determine its velocity at time t.*



Let PQ be the given curve.

Divide the time t into any number of intervals each equal to Δt, represented by OA_1, A_1A_2, A_2A_3 ... and draw the ordinates A_1a_1, A_2a_2 ....

Considering any interval A_3A_4,

the acceleration at its beginning is represented by A_3a_3,

" " " end " " " A_4a_4.

$\therefore$ the velocity generated in that interval lies between

$$A_3a_3 \times \Delta t, \text{ and } A_4a_4 \times \Delta t \qquad (v - u = ft)$$

i.e. this velocity is represented by an area which lies between the areas of the rectangles A_4a_3 and A_3a_4.

Similarly for each interval;

$\therefore$ the total change of velocity in time t is represented by an area between the sum of the smaller and the sum of the larger rectangles.

But when we make the number of intervals indefinitely large, the area of each set of rectangles approaches and ultimately coincides with the area bounded by PO, ON, NQ and the curve PQ.

$\therefore$ the change of velocity in time t is represented by the area $PONQ$.

Thus if v is the velocity at time t, and u the initial velocity,

$$v - u = \text{area } PONQ.$$

191. *If u is the initial velocity of a particle and its acceleration $= kt$ when k is constant, to prove that the velocity at time $t = u + \frac{1}{2}kt^2$.*

Draw the straight line OP, the graph of $f = kt$.

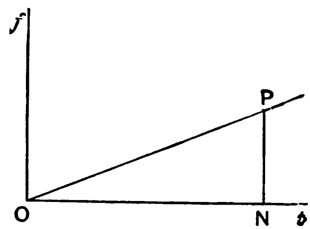
Then if $ON = t$, and PN is the ordinate at N,

$\triangle OPN$ represents the change of velocity in time t.

$\therefore$ if v is the velocity of the particle at time t,

$$v - u = \triangle ONP$$
$$= \tfrac{1}{2}PN \, . \, ON$$
$$= \tfrac{1}{2}kt \times t = \tfrac{1}{2}kt^2,$$

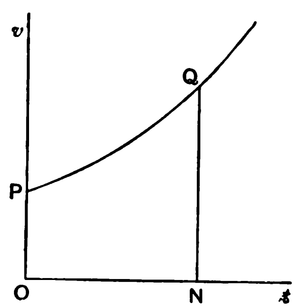
i.e. $$v = u + \tfrac{1}{2}kt^2.$$



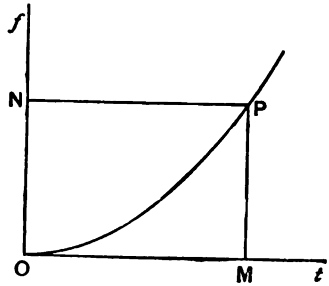
192. *Given the velocity-time curve of a particle to determine the space described in a given time t.*

If PQ is the given curve, by the method of Art. 190 it may be proved that area $PONQ$ represents the space required.

The actual proof is left as an exercise for the student.



193. *The acceleration f of a particle at time t is given by the equation $f = kt^2$, when k is constant: to find the velocity at time t, u being the initial velocity.*



The graph of $f = kt^2$ is a parabola OP, whose axis is the acceleration-axis, and whose vertex is at the origin.

$\therefore$ the change of velocity in time $t\,(OM)$ is represented by the area OMP.

$$\therefore\ v - u = \text{area } OMP$$

$$= \tfrac{1}{3} PM.OM \qquad \text{(Art. 190)}$$

$$= \tfrac{1}{3} kt^2 \times t.$$

$$\therefore\ v = u + \tfrac{1}{3} kt^3.$$

EXAMPLES XXII.

1. The acceleration f of a particle at time t is given by the equation $f = 3t$. Find its velocity at time t if it starts from rest.

2. A particle starts with a velocity of 3 ft. per sec. and moves under an acceleration which is $4t$ at time t. Find its velocity at time t.

3. A particle has a velocity $5t$ at time t. Find the space described in time t.

4. The velocity (in ft. per sec.) of a particle at time t is given by the equation $v = 3 + 4t$. Find the space described in 6 seconds.

5. At time t the velocity (in ft. per sec.) of a particle is $6t^2$: find the space described in 3 secs., and the acceleration of the particle.

6. The velocity, v, of a particle at time t is given by the equation $v = 4 + 6t^2$: find the space described in time t. Also find its acceleration.

7. If $2t+9t^2$ is the velocity of a particle at time t, find its acceleration and the space described in time t.

8. The co-ordinates (x, y) of a moving particle at time t, referred to rectangular axes are given by the equations $x=6t^2$, $y=8t^2$. Find the magnitude and direction of its velocity at time t, and prove that it is subject to a constant acceleration in a fixed direction. Find this acceleration.

9. The co-ordinates (x, y) of a moving particle at time t, referred to rectangular axes are given by the equations $x=3t+3t^2$, $y=4t+4t^2$. Prove that it is subject to a constant acceleration in a fixed direction, and find its velocity at time t.

MISCELLANEOUS EXAMPLES. XXIII.

1. A string passes over a fixed smooth pulley. To one end of it is attached a mass m, and to the other a smooth weightless pulley round which passes a string carrying at its ends masses which together make up M, the one being double of the other. Prove that if the system be in a vertical plane the acceleration of the string over the fixed pulley is what it would be if masses $8M$ and $9m$ were attached to its extremities, all other weights being removed.

2. A wedge of mass M, and angle a, can move freely on a smooth horizontal plane; a smooth sphere of mass m strikes it in a direction perpendicular to its inclined face and rebounds. Prove that the ratio of the velocities of the sphere just before and just after impact is $\dfrac{M+m\sin^2 a}{eM-m\sin^2 a}$, where e is the coefficient of restitution.

3. A string of length l is attached to a particle of mass m which slides on a rough horizontal plane, the other end of the string being attached to a fixed point in the plane. Initially the string is just tight and the particle is projected at right angles to the string with velocity u. Find the whole space described by the particle before coming to rest, and prove that the tension of the string when the particle has described a space s is equal to $\dfrac{m}{l}(u^2-2\mu gs)$, where μ is the coefficient of friction.

4. When a body projected horizontally reaches that point of its path where the horizontal and vertical components of its velocity are equal, prove that it has then descended through a vertical distance equal to that in which a body falling freely would acquire the velocity of projection.

5. A perfectly elastic ball is projected from a focus of an ellipse in any direction within the plane of the figure; shew that it will return, after two reflexions at the curve, to the same focus.

6. A shot fired from the ground, at an elevation of 45°, hits a mark half a mile off measured horizontally, and 140 ft. from the ground: find the velocity of projection.

7. AB is the vertical diameter of a circle. A perfectly elastic particle, after sliding down the smooth chord AC and being reflected by the plane BC, is allowed to move freely in space as a projectile. Shew that the body will strike the circle at the opposite extremity of the diameter CD.

8. If v be the velocity of projection, v_1, v_2 the velocities of a projectile when at heights h_1, h_2 respectively above the horizontal plane of projection, prove that $\frac{v^2-v_1^2}{v^2-v_2^2}=\frac{h_1}{h_2}$.

9. A given point lies in the same vertical plane with a circle, of radius a, at a greater height than any point of the circle; t_1, t_2 are the least and greatest times of descent down straight lines from the given point to any point of the circle: prove that $\frac{1}{t_1^2}-\frac{1}{t_2^2}=\frac{ga}{c^2-a^2}$, c being the distance from the given point to the centre of the circle.

10. A particle slides from rest down a smooth plane inclined at 30° to the horizon. Find the position of that length of 80 feet which is passed over by the particle in one second.

11. Two scale-pans, each of 5 oz. mass, are connected by a light inextensible string, which passes over a smooth fixed pulley. If a mass of 2 oz. be placed in one pan, and a mass of 4 oz. in the other, find the pressures of the masses on the scale-pans.

12. Three equal spheres whose coefficient of restitution is $\frac{1}{3}$ are ranged at rest in a straight line. The first sphere is then projected in the same straight line with velocity u, so as to impinge directly on the second. Find the final velocities of the three spheres after the last collision has taken place between them.

13. A particle is projected *in vacuo* from a point O with a velocity whose horizontal and vertical components are 64 ft. per sec. and 80 ft. per sec. (upwards) respectively. Draw a careful figure exhibiting the positions of the particle after 1, 2, 3, 4 and 5 seconds respectively, and state the coordinates of the particle at these instants referred to vertical and horizontal components through O. Find also the coordinates of the focus of the parabola described.

14. A stone is thrown from the top of a tower with a velocity of g ft. per sec. in a direction making an angle α with a line drawn vertically upwards through the point of projection; prove that at the end of 2 secs. the line joining the stone to the point of projection will make an angle $\frac{1}{2}(\pi+\alpha)$ with the vertical line.

15. A ball moving with a velocity of 1000 ft. per sec. has its velocity reduced by 400 ft. after penetrating one inch into a plank. Find how far it will penetrate into the plank before being stopped, assuming the resistance of the plank to be uniform.

16. A bullet is fired with a velocity of 1000 feet per second. Find the height (approximately) to which the bullet ascends when the horizontal range is one-tenth the greatest possible range.

17. A balloon when at a height of 3249 ft. from the ground begins to fall with a uniform acceleration $\frac{g}{16}$. When the balloon is at a

height of 1224 ft. from the ground, ballast to the amount of one-tenth the whole mass of the balloon is thrown downwards with a velocity, relative to the balloon, of 20 ft. per second. Find the time the ballast will take to reach the ground.

18. A stone is thrown from a height of 6 feet so as just to pass horizontally over a wall which is 29 yards high and 72 yards distant. Find the velocity and direction of projection.

19. If Q be the point in the path of a projectile where the tangent is at right angles to the tangent at the starting point P, the line through Q parallel to the tangent at P will meet the vertical through P at a distance $\frac{u^2}{2g\sin^2 a}$ below P, where u is the velocity, and a the angle of projection.

20. A particle is projected with velocity v at an inclination a to the horizontal, and at the instant it is moving horizontally, enters a smooth semicircular tube in the same vertical plane as the path described and having its diameter vertical and of length $2r$. Find the condition that it should, on emerging at the other end of the tube, return to the point of projection.

21. Heavy particles slide down chords of a vertical circle from rest at the highest point: prove that the locus of the vertices of the parabolic paths they describe after leaving the chords is an ellipse whose axes are in the ratio of 2 to 1.

22. A particle slides down a rough inclined plane AB; shew that if F be the focus of the parabolic path described after leaving the plane, the angle AFB

$$=\frac{\pi}{2}+\text{the angle of limiting friction.}$$

23. The edges of a groove cut in a smooth horizontal table are two concentric circles whose radii are $70\frac{1}{2}$ and $73\frac{1}{2}$ inches. A sphere of 4 lbs. mass whose diameter is 5 inches moves in this groove with uniform velocity. Find the greatest number of revolutions it may make per minute without leaving the inner edge, and if it make half this number, find the reactions at the points where it rests upon the edges.

24. A particle of mass m is projected from the centre of an imperfectly elastic ring of unit mass lying at rest upon a smooth horizontal table. Find m, so that from the third to the fourth impact the particle may be at rest.

25. A mass M hangs by a string over a pulley, a boy takes hold of the other end and climbs up a height h in t secs. without disturbing the mass. Investigate his motion and find his weight.

26. A body is projected along a rough horizontal plane with a given velocity. If at each point P of its path a vertical PH be drawn, such that its velocity at P is that due to a fall through the height PH, find the locus of H. Hence deduce a simple geometrical construction for determining the point at which it will stop.

27. A body is projected from A in a given direction AB. If, when the projectile is at P, Q be the point of AB vertically above P, and M be the bisection of AQ, prove that the direction of its motion is MP.

28. A particle is placed upon a rough horizontal plate ($\mu = \frac{1}{3}$) at a distance of 9 inches from a vertical axis about which the plate can turn: find the greatest number of revolutions per minute the plate can make without causing the particle to move upon it.

29. Within a smooth circular tube fixed in a vertical plane are two particles of masses P, Q connected by a string whose length is equal to half that of the tube. Find the acceleration of each particle in the direction of motion, and the tension of the string supposed tight, when the line joining the particles makes an angle θ with the horizon.

30. Two particles, projected with the same velocity from O, pass through the same point P; shew that if α, β be the angles of projection

$$\alpha + \beta = \frac{\pi}{2} + i$$

where i = angle OP makes with the horizon.

31. In Atwood's machine, two weights of 3 lbs. and 5 lbs. are attached to opposite ends of a string which passes over a light smooth pulley and a rider weighing 4 lbs. is placed over the smaller weight. When the system has moved through 18 inches from rest, the rider is detached by coming in contact with a fixed ring. How far will the 3 lb. weight descend below the ring before coming to rest?

32. A slip-carriage is detached from a train and brought to rest with uniform retardation in 3 minutes, during which time it travels $1\frac{1}{2}$ miles. With what velocity was the train travelling when the carriage was detached?

33. The weight of the whole train was 130 tons and that of the slip-carriage 10 tons. Before the carriage was detached the train was just kept going with uniform velocity by the pull of the engine, the resistance due to friction being 35 lbs. wt. per ton. Supposing the engine to pull the train with the same force after the carriage is detached, find the acceleration.

34. It is observed on board a steamer steaming due North that the direction of the smoke is α West of South, and when steaming due West the direction is β East of North: find what the direction of the wind is, and what direction the smoke would take if the steamer turned her head due East.

35. A uniform rectangular block stands on its base AD on a rough floor. It is pulled at C by a horizontal force just great enough to begin to turn it round the corner D. If this same force continues to pull it horizontally at C till the block has turned through an angle θ, and then ceases, prove that the block will have acquired sufficient momentum to cause it just to overturn round D, provided $\sin\theta = \tan\frac{a}{2}$ where a is the angle BDC.

36. A heavy slab, of uniform thickness, whose under surface is rough, but the upper smooth, slides down a given inclined plane. Find the acceleration with which a particle laid on its upper surface will move along the slab.

37. If the force of attraction, for different distances, varies inversely as the squares of the distances, and for different bodies, directly as their masses, prove that if several bodies move round a point in concentric circles, the squares of the times of revolution are as the cubes of the radii.

38. Find the greatest range which a projectile with an initial velocity of 1600 ft. per sec. can attain on a horizontal plane. Shew also that for a small difference of elevation not exceeding 10 minutes, whether of excess or defect, the range attained will fall short of the maximum by less than $1\frac{1}{2}$ feet.

39. If a be the distance between two moving points at any time, V their relative velocity, and u, v the resolved parts of V in, and perpendicular to, the direction of a, shew that their distance when they are nearest to each other is $\frac{av}{V}$, and that the time of arriving at this nearest distance is $\frac{au}{V^2}$.

40. A hill in the form of a right circular cone rises out of a plain. A man wishes to drag a weight slowly from one point of the plain at the base of the hill to the diametrically opposite point of the base. Prove that the work he will do if he drag the weight round on the plain skirting the hill is to the work he will do if he drag the weight straight over the top of the hill in the ratio of π to 2; where the coefficient of friction of the weight with the ground is assumed the same at all points of the hill and plain, and the weight can rest on the hill without sliding.

41. A small pulley of mass M is lying on a smooth table; a light string passes round the pulley and has weights of masses m and m' attached to its ends, the two portions of the string being perpendicular to the edge of the table and passing over it so that the weights hang vertically; prove that the pulley moves with acceleration

$$\frac{4mm'g}{M(m+m')+4mm'}.$$

42. Two masses m and $2m$ are connected by an inextensible string which passes over a fixed smooth pulley; find their velocity at the end of three seconds. Also determine the instantaneous effect, and the subsequent change of motion produced, by suddenly attaching at that time a mass m to the ascending body.

43. A ball of elasticity $\frac{1}{2}$ falls from a height of 64 ft. upon a horizontal plane; find the height to which it will rise at the first rebound, and the time at which the rebounding will cease.

44. A mass of 1 lb. is suspended by a string 2 ft. long, in a railway carriage. Shew that when the train is moving round a curve, whose radius is 22 chains, at the rate of 30 miles an hour, the tension of the string is increased by about $\frac{1}{72}$ oz., and the horizontal displacement of the weight is 1 in. nearly.

45. The trail of smoke from a steamer on a course due north is observed to extend in the direction E.S.E., while that from another, on a course due south, with the same uniform speed, is observed to be N.N.E.; determine the speed and direction of the wind.

46. A ball impinges on another ball of equal mass at rest, and after impact the directions of motion of the two balls make equal angles (θ) with the direction of motion of the first ball before impact: prove that $\tan\theta=\sqrt{e}$, where e is the coefficient of restitution.

47. A wedge of mass M rests with a rough face in contact with a horizontal table, and with another face, which is smooth, inclined at an angle a to the table. The angle of friction between the wedge and the table is λ. A particle of mass m slides down the smooth face. Find the condition that the wedge may move; and prove that, if it move, its acceleration is

$$\frac{m\cos a\sin(a-\lambda)-M\sin\lambda}{M\cos\lambda+m\sin a\sin(a-\lambda)}g.$$

48. Two elastic spherical projectiles, of masses M and m respectively, impinge directly. The axis of the parabolic path of M is shifted through a distance D, and that of the parabolic path of m through a distance d: also the energy lost by M is E, and that gained by m is e. Prove that

$$\frac{D}{d}=\frac{mE}{Me}.$$

49. A string has attached to its extremities masses each equal to M, and is then passed over two small pulleys whose centres are in a horizontal line, and at a distance $2a$ apart. A mass $2M$ is attached midway between the pulleys and is then let go; shew that the subsequent velocity of the mass $2M$

$$=2\sqrt{ag}\sqrt{\frac{1-\tan\frac{\phi}{4}}{3+\cos\phi}},$$

where ϕ is the angle between the two parts of the string.

50. A small billiard ball is at the centre of a rectangular billiard table whose sides are of lengths a and b. Another equal ball is placed at such a point of one of the sides a that when projected in a proper direction it will strike the other ball and drive it into one of the pockets at the corners, whilst it goes itself into the pocket in the middle of the opposite side a. Prove that the second ball must be placed at a distance from the centre of the side which it touches equal to

$$\frac{(1+e)\,ab^2}{2[(1-e)a^2+2b^2]},$$

where e is the coefficient of restitution.

51. Two straight railways converge to a level crossing at an angle α; and two trains are moving towards this point with velocities u, v, respectively. Find when they are nearest to one another, and prove that their least distance apart is

$$\frac{(av \sim bu)\sin\alpha}{\sqrt{u^2+v^2-2uv\cos\alpha}},$$

where a and b are the initial distances of the trains from the level crossing.

52. From a fort, of height H, a shell is thrown with velocity due to a height h. Shew that the greatest range on a horizontal plane through the foot of the fort is $\sqrt{4h(H+h)}$, and that the angle of projection corresponding to this range is

$$\cos^{-1}\sqrt{\frac{H+h}{H+2h}}.$$

53. Two particles of equal mass are connected by an inextensible string of length l, and lie at rest on a smooth horizontal table with the string straight. A horizontal blow is given to one particle in a direction perpendicular to the string. Prove that, when next the string is parallel to its initial direction, it lies in a line distant $\frac{1}{2}\pi l$ from its initial position.

54. Two spheres of equal mass m, moving towards each other with the same velocity u, collide simultaneously with a stationary sphere of mass M, so placed that at the moment of collision the line joining the centres of the moving spheres subtends a right angle at the centre of the third. Supposing the latter free to move, prove that it will start with velocity

$$\frac{mu(1+e)}{M+m},$$

where e is the coefficient of elasticity.

55. If a gun be sighted for a given range with given muzzle velocity and be moveable about its muzzle in a vertical plane; shew that it is accurately sighted for all points lying on a certain parabolic arc.

56. Two smooth beads of equal weight are threaded on a light inextensible string which is fastened to two points at the same level. If the beads move symmetrically towards each other in a vertical plane, starting from rest in the position in which the terminal portions of the string are vertical, shew that each bead describes a parabola.

57. Two particles of equal mass m are made fast to the ends of a light inextensible string passing over two smooth pegs $2a$ ft. apart in the same horizontal line. A mass M $(< 2m)$ is then attached to the string midway between the pegs and is allowed to fall; find how far it will descend before coming to rest.

58. A smooth pulley of mass M_1 is laid on a fixed smooth inclined plane of inclination a, over it passes a string to the ends of which are attached particles of masses m_1, m_2 respectively, which are otherwise free to move on the plane. The pulley is connected with a particle of mass M_2 at the end of a string which passes over a smooth pulley at the top of the plane and hangs vertically. Find the space traversed in one second by the particle M_2.

59. Two smooth elastic spheres of given masses are moving in perpendicular directions and come into collision so that the line joining their centres makes an angle of 45° with the direction of each of them; shew that if they move off in parallel directions after impact, their kinetic energy before impact was that of a particle whose mass is the arithmetic mean of their masses, and velocity the geometric mean of their original velocities, the coefficient of elasticity being $\frac{1}{3}$.

60. A railway engine is moving horizontally round a curve. If the radii (r_1, r_2) of the inner and outer rails, which are at the same level, and the height (h) of the centre of gravity of the engine, be given; find the greatest velocity that the engine can attain without tilting up.

61. If a string having masses m and n attached to its ends pass over a fixed weightless pulley, then under a moveable pulley of mass M, and again over another fixed weightless pulley, shew that the acceleration of M is

$$\frac{M(m+n)-4mn}{M(m+n)+4mn}g,$$

the portions of string all being vertical.

62. A ball moving uniformly in a straight line with velocity u, meets a small inclined plane whose intersection with the horizontal plane is perpendicular to the direction of the ball's motion; shew that after a time $\frac{u}{g}\cdot\frac{1+e}{1-e}\sin 2a$ the ball will again be moving uniformly along the horizontal plane with a velocity $u(\cos^2 a - e\sin^2 a)$, where e is the coefficient of elasticity of both planes and a the inclination of the small plane.

63. A particle is projected from a point at a height c above a horizontal plane with velocity $\sqrt{gc}$; shew that the farthest point on the plane that the body can reach is at a distance $2c$ from the point of projection.

64. A smooth wedge, whose angle is a, has one face in contact with a horizontal plane. Find the acceleration with which it must be made to move that a heavy particle may be in relative equilibrium on its inclined surface.

65. Two bullets of masses m, m', which are describing parabolas of latera-recta l, l' in the same vertical plane, collide and coalesce. Prove that the latus-rectum of their path after impact will be

$$\left[\frac{m\sqrt{l} \pm m'\sqrt{l'}}{m+m'}\right]^2.$$

66. A large number of equal particles are fastened at unequal intervals to a fine string, and then collected into a heap at the edge of a smooth horizontal table with the extreme one just hanging over the edge; the intervals are such that the times between successive particles being carried over the edge are equal; prove that if c_n be the interval between the nth and $\overline{n+1}$th particle and v_n the velocity just after the $\overline{n+1}$th particle is carried over

$$\frac{c_n}{c_1} = \frac{v_n}{v_1} = n.$$

67. If a be the penetration of a shot of m lbs. striking a fixed iron plate with velocity v, shew that an iron plate of M lbs. and thickness b, free to move, will be perforated if

$$b < \frac{Ma}{M+m}.$$

68. A particle hangs from a fixed point in a wall by a string of length a, and is projected horizontally with velocity u. If the string come in contact with a nail in the wall situated in the horizontal line through the point of suspension and at a distance b from it, find the least value of u in order that the particle may make a complete revolution round the nail, without the string becoming slack.

69. Two equal and perfectly elastic spherical beads, each of radius r, strung upon an inextensible string, are placed on a smooth table and are drawn apart to the greatest possible distance a between their centres. One of them is then projected directly towards the other with a given velocity u. Investigate the motion and determine their velocities, at any time t after projection.

70. Two equal balls A, B are lying very nearly in contact on a smooth horizontal table. A third equal ball impinges directly on A, the three centres being in the same straight line; prove that if

$e > 3 - 2\sqrt{2}$, B's final velocity will bear to the initial velocity of the striking ball the ratio $(1+e)^2 : 4$.

71. A gun of mass M, with its barrel horizontal, fires a shot of mass m. When the gun is fixed the shot strikes a vertical wall in front of the gun at a depth h below the barrel. Prove that, if the gun be free to move on a horizontal plane, the shot will strike the wall at a depth $\frac{mh}{M}$ below the other point of impact. [Assume that the total energy wasted in generating heat, etc. is the same in each case.]

72. A particle is projected from a point in a horizontal plane so as to strike a vertical wall at right angles, and after rebounding from the wall, and once from the horizontal plane, to pass through the point of projection. Prove that the elasticity of the particle is $\frac{1}{2}$.

73. One end of a uniform chain is held above a fixed inelastic horizontal plane; part of the chain hangs vertically and the rest is coiled up on the plane: the upper end of the chain is then released, and the vertical part falls; shew that at any time the pressure on the table is less than three times the weight of chain lying on the table at that time, by twice the weight of chain originally lying there.

74. Shew that the kinetic energy of two masses is equal to the kinetic energy of the sum of two masses moving with the velocity of the centre of inertia together with that of each mass moving with its velocity relative to the centre of inertia.

75. A railway train passes from one station to another a miles distant, starting with the uniform acceleration f, and when steam is shut off and the brake applied, slowing up with the uniform retardation f'. Find the time taken, f and f' being expressed in ft.-sec. units.

76. If two bodies start at the same instant sliding down two lines in the same vertical plane sloping towards the same direction, at angles α and β to the horizon, prove that each as seen from the other will always appear to be moving parallel to a line inclined to the horizon at the angle $\alpha+\beta$.

77. A circus horse gallops round a circle of 30 ft. radius, at a speed of 15 miles an hour; prove that the least value of the coefficient of friction between feet and ground, that the horse may not slip, neglecting the distance between the feet and centre of gravity, is $\frac{1}{2}$ very nearly.

78. Two masses P and Q $(P > Q)$ are suspended by a light string over a pulley of inappreciable mass. After moving for a time t, a mass $P-Q$ is instantaneously attached to the ascending mass Q. Find the whole motion of the system.

79. Two men, weights W and $W+w$, starting simultaneously from the ground, swarm, with uniform vertical accelerations, up the two free

lengths of a weightless inextensible rope, which passes over a smooth pulley, at the height h from the ground. If the lighter of the two men reach the pulley in t secs., prove that the heavier cannot get nearer to it than

$$\frac{w}{W+w}\left(\frac{gt^2}{2}+h\right).$$

80. Projectiles are discharged in vacuo from a given point with a constant elevation. Find the locus of the vertices of the parabolas which they describe.

81. A shot is fired with a given velocity from the point A up an inclined plane, so as to have a maximum range, and strikes the plane at the point A'. With what velocity and elevation must it be fired back from A' so as to return to A by the same course?

82. A body projected from a point in the circumference of a circle returns to the point it started from after two reflexions from the circumference. Supposing the coefficient of elasticity to be $\frac{9}{10}$, find the angle between the line along which the body first starts and that on which it returns to the starting point.

83. A rigid square $ABDC$ composed of four smooth wires is fixed so that A is vertically above D. Two small equal spherical beads (of elasticity e) slide down BD, CD starting simultaneously from B and C. Shew that their velocities of approach and separation at D are in the ratio of 1 to e, and that after impact they will separate till the distance between them is $e^2 . BC$.

84. Straight lines are drawn in a vertical plane such that the time of quickest descent from each line to a given circle in the plane is the same for all. Shew that the straight lines touch a fixed circle.

85. A particle is projected at a given angle with the horizon to strike a given inclined plane, not passing through the point of projection; find the time of flight. Shew that it is least (for a given velocity of projection) when the particle is projected perpendicularly towards the plane.

86. Three imperfectly elastic equal and similar balls, A, B, C, whose centres are in one straight line, impinge on one another, B and C being initially at rest and A in motion. Shew that there must be at least three impacts; and that there will be only three if the coefficient of restitution lies between $3-2\sqrt{2}$ and unity.

87. A particle is projected along the circumference of a smooth vertical circle of radius a. It starts from the lowest point with velocity $\sqrt{\frac{7ga}{2}}$; shew that it leaves the circle when the radius to it from the centre makes an angle of 30° with the horizontal and that it then describes a parabola which passes through the point of projection.

88. In an Atwood's machine, to the smaller mass is attached a string, which, passing under a smooth pulley fixed on a table vertically beneath, is fastened to a third mass on the same table; the coefficient of friction being μ, find the tensions of the strings, if the mass on the table begins to move.

89. A perfectly elastic body is projected at an angle α to the horizon, up an inclined plane of angle β. Shew that if

$$\tan\alpha = \cot\beta + 2\tan\beta,$$

it will return to its original position after one bounce.

90. Three particles of masses m, m, m', and three light strings of lengths $\frac{2\pi a}{3}$ connecting them, lie in a vertical smooth circular tube of radius a: if the particles be placed in that position of equilibrium which is unstable and slightly displaced, find the greatest velocity of their motion.

91. ABC is a right angle. If C begins to move with angular velocity β about B, BC remaining constant, and B begins to move with angular velocity α about A, AB remaining constant, shew that the initial velocity of C about A is

$$\alpha\cos^2 A + \beta\sin^2 A.$$

92. Over a smooth light pulley is passed a string, supporting at one end a weight of mass 6 lbs., and at the other a smooth pulley of mass 1 lb. A string with weights whose masses are 2 lbs. and 3 lbs. is passed over the second pulley; prove that the velocity of the 2 lb. mass at the end of two seconds will be $\frac{26g}{59}$.

93. Two spheres of masses m, m' move in parallel lines at a small distance h apart, and the former overtakes the latter when their velocities are u, u'. Prove that, neglecting h^2, after collision the former sphere has a velocity

$$\frac{m'h(1+e)(u-u')}{c(m+m')}$$

perpendicular to the former line of motion, where c is the sum of their radii and e the coefficient of elasticity.

94. A particle of mass m is attached by a string to a fixed point C, and by another string to a smooth ring of mass m' which can freely revolve round and slide along a vertical rod passing through C, both strings being weightless and inextensible. If the lengths of the vertical projections of the strings are x and y when the whole system is revolving with constant angular velocity ω about the rod, prove that

$$\frac{m\omega^2}{g} = \frac{m+m'}{x} + \frac{m'}{y},$$

the ring being so small that its angular momentum may be neglected.

95. Two inelastic balls of equal size, but of masses m, m', lie in contact on a smooth table. That of mass m receives a blow in a direction through its centre making an angle a with the line of centres. Shew that the kinetic energy is

$$\frac{mm'+m'^2 \sin^2 a}{mm'+m^2 \sin^2 a}$$

of what it would have been if the balls had been interchanged, and m' had received the blow.

96. An elastic ball is projected along a straight horizontal tunnel, from the level of the floor, so as to strike the roof, then the floor, and so on. If the vertical height of the tunnel be h, the vertical velocity of the ball be due to a height H, and the coefficient of elasticity be e, shew that the ball will strike the roof of the tunnel n times, where n is the integer next greater than

$$\tfrac{1}{4} \log \left[\frac{H(1+e^2)-he^2}{h}\right] \div \log \frac{1}{e}.$$

97. Two points P and Q describe in the same direction two co-planar concentric circles (centre O) with velocities p and q, the radii of the circles being a and b; if $p:q=\sqrt{b}:\sqrt{a}$, prove that when the angular velocity of PQ vanishes, the angle

$$POQ=\cos^{-1}\left(\frac{\sqrt{ab}}{a+b-\sqrt{ab}}\right).$$

98. A particle moving on a smooth horizontal plane strikes a rough vertical wall; shew that if a, a' are the angles the directions of motion make with the wall before and after impact respectively, then

$$e \tan a' = \tan a - \mu(1+e).$$

What happens when $\mu(1+e) > \tan a$?

99. A string, fixed at one end, passes under a smooth moveable pulley of mass 4 lbs., then over a smooth fixed pulley, and has a mass of 4 lbs. attached to its free end; find the acceleration with which this mass moves, the different portions of the string being parallel.

100. A particle is projected in a vertical plane from a given point in a given direction, so that its path touches a given straight line in the same plane. Find, by a geometrical construction, its point of contact.

101. A wedge of mass M can slide on a smooth horizontal plane. The wedge has a smooth face inclined at an angle a to the horizontal. Initially the wedge is at rest, and a particle of mass m is projected directly up its inclined face. Prove that, if the particle rises to a height h above its point of projection, its velocity of projection is

$$\left\{2gh\frac{M+m}{M+m\sin^2 a}\right\}^{\frac{1}{2}}.$$

102. In the first system of pulleys, in which a string passing round each pulley has one end attached to a fixed beam and the other to the pulley next above, there is no 'power' and no 'weight.' The n moveable pulleys are all of equal weight, they are smooth, and can all be treated as particles in calculating their motions. The string is without mass. Prove that the acceleration of the lowest pulley is $\frac{3g}{2^n+1}$.

103. On a string of length $2na$ are strung $(2n+1)$ equal particles, at equal intervals a. The string is placed symmetrically over a smooth fixed pulley, whose radius is small compared with a, and being slightly displaced, is allowed to move freely under gravity. Shew that the string leaves the pulley with a velocity given by

$$v^2=\frac{2gan^2}{2n+1}.$$

104. Three thin smooth tubes form a triangle ABC in a vertical plane, BC being horizontal and A upwards. Particles of masses m_1, m_2 start from A and slide down the sides AB, AC under gravity. Find the path of the centre of mass, G, of the particles, and the acceleration of G.

105. A particle of given elasticity is projected in a horizontal plane from a given point A, and after reflexion at two given vertical planes passes through a given point B; find, by geometrical construction, its points of impact on the planes.

106. If the unit of energy be the energy of a pound which has fallen through 100 ft. from rest, and the unit of time be the time it has taken to fall, while the unit of length is 5 ft., find the unit of mass.

107. A mass attached to one end of a light inextensible string which passes over a smooth pulley is descending with uniform velocity; to the other end of the string is attached a ring without mass, through which passes a second string supporting at its extremities masses m and m'. If one of these masses on arriving at the ring become entangled in it, shew that it will afterwards move with acceleration

$$\frac{(m-m')^2 g}{m^2+m'^2+6mm'}.$$

108. Two equal particles are joined by a light inextensible string of length l, which is stretched out perpendicular to the edge of a smooth horizontal table. One particle is gently pushed over the edge; shew that the whole string will next be horizontal at a depth $(\pi^2+2\pi+4)\frac{l}{8}$ below the surface of the table.

109. Two small rings of masses m and m' $(m>m')$ are capable of sliding on a smooth circular wire of radius a, whose vertical

diameter is fixed, the rings being connected by a light string of length $a\sqrt{2}$; prove that, if the wire is made to rotate round the vertical diameter with an angular velocity

$$\left\{\frac{2g}{a\sqrt{3}}\cdot\frac{m\sqrt{3}-m'}{m-m'}\right\}^{\frac{1}{2}},$$

the rings can be in relative equilibrium on opposite sides of the vertical diameter, the radius through the ring m being inclined at an angle 60° to the vertical. Shew also that the tension of the string is

$$\left(\frac{mm'}{m-m'}\right)\cdot\frac{\sqrt{3}-1}{\sqrt{2}}g.$$

110. A spherical ball, whose centre is at A, is projected so as to pass through a hoop of diameter equal to that of the ball, the centre (B) of the hoop being above the horizontal plane through A, and the plane of the hoop being perpendicular to the vertical plane through AB and inclined at an angle a to the horizon. Find the direction of projection in terms of a and the coordinates of B, and prove that the time before passing through the hoop is that of a particle sliding from rest down a line equal and parallel to the projection of AB upon the plane of the hoop.

111. Prove that the work done in slowly extracting a cork of length l from the neck of a bottle is $\frac{\mu Pl^2}{2}$, if the total normal pressure per unit length between the part of the cork unextracted at any instant and the bottle be constant and equal to P, and the coefficient of friction be μ.

112. The shape of an open umbrella is a paraboloid of revolution (the surface generated by the revolution of a parabola about its axis). If it be made to revolve about the stick, held vertically, with angular velocity ω, prove that the latera recta of the paths described by drops of rain flying off it are proportional to the original depths of those drops below the vertex.

113. A body projected from the top of a tower at an angle ϕ with the horizon, after t secs. strikes the ground at a distance a from the foot of the tower; find the velocity of projection, and the height of the tower.

114. Find the velocity and angle of projection of a particle which being thrown from the level of the ground just clears a wall 14 ft. high at a distance of 40 ft. from the point of projection, and strikes a wall parallel to the former and 20 ft. beyond it, at a point 9 ft. above the ground, the plane of projection being perpendicular to the walls.

115. A particle of mass 8 lbs., attached by a string of length 6 ft. to the vertex of a smooth cone whose axis is vertical and semi-vertical

angle 30°, describes a horizontal circle on the surface of the cone with uniform velocity 6 ft. per sec.; find the tension of the string and the pressure on the surface.

116. Two lines ORP, OQ in a horizontal plane make an angle a with one another. An elastic particle is projected in the horizontal plane from P, strikes OQ at Q, and returns to R, where $PR=n\,.\,OR$. Prove that the angle $OPQ\,(=\theta)$ of projection is given by the equation

$$\cot\theta=\tan a+\frac{2en}{(1+e)\sin 2a},$$

e being the coefficient of elasticity.

117. Two spheres are approaching one another, their centres moving in one and the same straight line; their masses are 8 ozs. and 4 ozs., their respective velocities 30 and 50 ft. per sec., and their coefficient of elasticity is $\frac{1}{2}$; find their velocities after collision. If they are in contact for $\frac{1}{64}$ of a second, find, in ounces weight, the magnitude of the mean pressure between them.

118. The centres of two equal billiard balls, of radius a and elasticity $\frac{1}{3}$, move along the straight lines whose equations are, in rectangular coordinates, $x-y-a=0$ and $x+a=0$, in such a manner that the line joining them is always parallel to the axis of x, and impinge at the origin. Find the equations of their lines of motion after impact.

119. A heavy particle attached to a fixed point by a light string makes complete revolutions in a vertical plane under the action of gravity. If the ratio of its maximum to its minimum velocity be 7 : 4, shew that the maximum tension of the string will be to the minimum in the ratio of 229 : 31.

120. A bullet is fired in the direction towards a second equal bullet which is let fall at the same instant. Prove that the two bullets will meet and that, if they coalesce, the latus rectum of their joint path will be one-quarter of the latus rectum of the original path of the first bullet.

121. Two bodies of equal mass hang from the ends of a light string which passes round a smooth fixed peg, and when at rest are 8 ft. and 19 ft. above a fixed horizontal plane. A piece of the upper body breaks off and strikes the plane at the same moment that the lower body does. Find the ratio of the parts into which the upper body is divided.

122. Two billiard balls P, Q of diameter 2 in. stand upon a smooth table $ABDC$, their centres being at distances 6 in. and 5 in. from the cushion AC and 4 in. and 6 in. from the cushion AB. Find the direction in which P must be struck so that after impinging in succession upon AB, AC it may strike Q directly, taking $\frac{1}{2}$ to be the coefficient of elasticity of the indiarubber cushion.

123. A man who can throw a stone up a height of 121 ft., goes down a coal-pit at the rate of $3\frac{3}{4}$ miles per hour. When he is $106\frac{82}{256}$ ft. from the top, he throws a stone up the pit-shaft. Shew that it will pass him in $2\frac{59}{64}$ secs. after it reaches the top: and find the depth of the pit if he has still $797\frac{217}{256}$ ft. to descend, when the stone reaches the bottom.

124. A (mass m) and B (mass m') are two particles connected together by a stretched string of length $2a$, lying on a smooth table so that AB is perpendicular to its edge, and A is beginning to fall off. Prove that in time

$$\frac{\pi+4}{2}\sqrt{\frac{a}{g}\,\frac{m+m'}{m}}$$

the particles will have fallen the same distance.

125. In the system of pulleys in which the same string goes round all the pulleys, and in which the power is replaced by a weight w, and W is the weight, it is observed that w ascends 7 ft. when W descends 1 ft.; again, it is observed that at a certain instant, W is descending with a velocity of 1 ft. per sec., and that after 1 sec. more it is descending with a velocity of 5 ft. per sec. Find the relation between w and W.

126. A number of bodies are dropped simultaneously from the heights m^2, n^2, etc. feet above a perfectly elastic horizontal plane, where m, n, etc. are whole numbers. Prove that, if g be taken as 32 on the usual assumption as to units, the whole system will be in its original position after successive intervals of $\frac{M}{2}$ seconds, where M is the least common multiple of m, n, etc.

127. A rectangular prism of triangular base rests with one of its faces on the ground, the other faces making angles α and α' with the horizon; and two particles of weights w and w' are placed on these two faces, connected together by a fine string passing over the edge of the prism so that the string is stretched tightly and lies in a vertical plane. The particles begin to slide; find their acceleration, and the force which is necessary to apply to the prism to keep it from moving, everything being regarded as quite smooth.

128. Two elastic spheres of masses m_1, m_2 impinge obliquely; v_1, v_2 are their velocities at any time. It being given that $m_1v_1^2+m_2v_2^2$ is the same after impact as before, prove that the velocity of separation is equal to that of approach.

129. A particle is projected from a point at the foot of one of two parallel vertical smooth walls so as, after three reflexions at the walls, to return to the point of projection, the third impact being direct; prove that $e^3+e^2+e=1$, and that the vertical heights of the three points of impact above the point of projection are as $e^2:1-e^2:1$.

130. A particle is projected up a rough inclined plane with a velocity due to falling vertically from a point A above the point of projection, and through A a straight line is drawn at an angle λ to the horizon, λ being the angle of friction; shew that this line will meet the inclined plane at the point where the particle comes to rest.

131. Three equal billiard balls A, B, C are lying on a billiard table; A and B are in contact, and equidistant from C. The ball C, projected *very nearly* in the direction of the point of contact, strikes A first and B *immediately* afterwards. Shew that after impact the velocities of A and B are in the ratio of 4 to $3-e$, e being the coefficient of restitution.

132. Three strings are knotted together at C. One string passes round a smooth peg A and supports a weight P at its free extremity; a second passes round a smooth peg B in the same horizontal line as A, and supports a weight Q; the third hangs vertically and supports a weight R. Prove that the work done in carrying C from its position of equilibrium to the peg A is $2Q \, . \, c \sin^2 \dfrac{\theta}{2}$, where $AB=c$, and θ is the angle which BC makes with the horizon in the position of equilibrium.

133. A gun is fired when it is moving forwards with a velocity of 6 ft. per sec., and the recoil brings it to rest. The mass of the gun and carriage is 100 tons, of the shot 1000 lbs. and the mass of the powder may be neglected. Find the velocity with which the shot leaves the gun, and the mean pressure on the base of the shot, supposing it to take $\frac{1}{32}$ sec. in moving through the bore.

The shot strikes a suspended target whose mass is 50 tons; find the velocity with which it begins to move after impact, supposing (1) the shot to be imbedded in it; (2) that the shot just gets through.

134. Three balls A, B, C, of equal mass and size, and moving with the same velocity v in directions inclined at 120° to one another, impinge so that their centres are at the angles of an equilateral triangle. If the coefficient of restitution between C and A or B be e, and between A and B be e', shew that A and B separate with a velocity $\dfrac{2e'+e}{\sqrt{3}} v$, it being assumed that compression ends at the same instant for all three balls.

135. Two equal weights of 3 lbs. are hanging close together, attached to the ends of a flexible string 6 yds. long, which passes over a smooth pulley 4 yds. above an inelastic floor. A weight of 1 lb. is attached to the string at a point 1 yd. above one of the equal weights, and the system set free. Trace the whole of the subsequent motion.

136. A ball is projected from a point in the floor of a room of height h, with velocity v and elevation θ, in a vertical plane perpen-

4. A body is dropped from a balloon moving horizontally with a velocity of 32 ft. per sec., and reaches the ground in 10 secs.; find the height of the balloon and the velocity of the body on striking the ground.

5. A heavy elastic ball drops from the ceiling of a room and, after twice rebounding from the floor, just reaches a point half the height of the room. Shew that its coefficient of elasticity is $\sqrt[4]{\frac{1}{2}}$.

6. An engine is travelling at thirty miles an hour; find the angular velocity, at any instant, of the highest point of one of the wheels (diameter 6 ft.) round its point of contact with the rails.

7. Determine the velocity and angle of projection of a projectile, so that it may strike a given mark just when it is moving horizontally.

Paper II.

1. A 3-ton cage, descending a shaft with a speed of 9 yards a second, is brought to rest by a uniform force in the space of 18 ft. Find the value of the force in tons weight.

2. Two equal masses (m) connected by a light inextensible string hang over a smooth pulley. Another equal mass is now laid upon one of them: find the pressure between these two masses during the ensuing motion.

3. A balloon is 432 yards high and is moving horizontally with the wind at 4 miles an hour. Find the path of a body dropped from the balloon, (1) relative to the balloon, (2) in fixed space. When does the body reach the ground?

4. One particle moves in a straight line with an acceleration $f \cos \alpha$, and a second particle in a line perpendicular to the former with an acceleration $f \sin \alpha$; if their initial velocities are u and v respectively, find their relative velocity at any time t.

5. The resolved parts of the velocities at the extremities of any chord of the parabola described by a projectile, at right angles to the chord, are equal.

6. A heavy particle slides down a smooth inclined plane on a given base a; find the height of the plane when the time of sliding down it is a minimum.

Paper III.

1. A 12-stone man is in a lift. Find his pressure on the floor of the lift, (1) when it is ascending with a uniform velocity of 8 ft. per sec., (2) when it is ascending with an acceleration of 8 ft.-sec. units.

2. A railway carriage runs freely down a smooth incline. Prove that if a weight be suspended by a string from the roof, it will hang with the string perpendicular to the floor. If the angle of the incline be a, and the weight be W, find the tension of the string.

3. If a body be projected horizontally with a velocity of 500 yds. per second, when will its path be inclined at an angle of 45° to the horizon? Shew that when the body attains this position it will have travelled horizontally twice as far as it has fallen vertically.

4. Determine the ratio of the masses of two perfectly elastic balls so that one, after impinging directly on the other at rest, may lose $\frac{1}{n}$th of its velocity.

5. A ship steaming due north with velocity v observes a light which appears to continue in the N.W., but always to be drawing nearer. Supposing it to proceed from another vessel steaming at the same rate, find the direction in which she is really moving, and the rate at which the ships are approaching one another.

6. A man shoots at a bird at a distance a due north of him, flying due east with velocity u; how far ahead of the bird must he aim so as to hit it, if v be the velocity of the shot?

Paper IV.

1. A body weighing w lbs. is acted on by a horizontal force which would support a body of $3w$ lbs. at rest. In what time will it acquire a velocity of v ft. per second?

2. A body is suspended by a cord from the roof of a railway carriage forming part of a railway train, and it is observed that the cord is inclined to the vertical at an angle whose tangent is $\frac{1}{10}$. In what time will the train acquire a velocity of sixty miles an hour?

3. At a given instant a particle is sliding down a rough plane, inclined at an angle $\sin^{-1}\frac{5}{13}$ to the horizon, with a velocity of 16 ft. per second; if the coefficient of friction be $\frac{3}{8}$, how much further will it move before coming to rest?

4. A ball is projected and a second ball also from the same point, and in the same direction, with a velocity equal to the vertical velocity of the first ball. Prove that the path of the second passes through the focus of the path of the first.

5. A ball 5 ozs. in weight falls from a height of 20 ft. on a fixed horizontal plane, and on rebounding reaches a height of $11\frac{1}{4}$ ft.; find the coefficient of elasticity, the measure of the impulse, and the distance travelled by the ball before it finally comes to rest.

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6. If the level of the water in docks covering 100 acres be altered 20 feet by the tide twice every 25 hours, find the horse-power of an engine which could be driven by the energy, if it could all be made use of.

Paper V.

1. A $\frac{1}{2}$ oz. bullet is fired with a muzzle velocity of 2000 ft. per sec. from a smooth gun having a barrel which has a length of 3 ft. and an internal diameter $\frac{3}{10}$ inch. Assuming that the pressure in the barrel is constant, find its measure in lbs. wt. per square inch.

2. ABC is a right-angled triangle in a vertical plane with AB horizontal; if PB be the line of quickest descent from the hypotenuse AC to the point B, find AP.

3. AC, BC are two inclined planes of elevation α and β respectively; a heavy inelastic particle slides down the plane AC from the point A; find how far it ascends the plane CB, when $CA=b$.

4. Two ships sail uniformly in the directions AB, AC, with velocities u, v, respectively. The former is at A when the latter is at C. The angle CAB, which is less than 90°, is equal to θ, and u is greater than v. Shew that if $AC=a$, the least distance of the ships apart will be

$$\frac{au\sin\theta}{(u^2+v^2-2uv\cos\theta)^{\frac{1}{2}}}.$$

5. A ring, of radius a, rolls upon the concave side of the circumference of a fixed ring of radius ma, the point of contact having the velocity v. Prove that the angular velocity of the moving ring is

$$\frac{m-1}{m}\cdot\frac{v}{a}.$$

If $m=2$ prove that every point on the moving ring describes a diameter of the fixed ring.

6. A particle (whose coefficient of restitution is e) is projected at right angles to a plane inclined at an angle α to the horizon with a velocity v. Find the latera recta of the successive paths, and shew that, before ceasing to rebound, it will have described along the plane a distance

$$\frac{2v^2\sin\alpha}{g\cos^2\alpha(1-e)^2}.$$

Paper VI.

1. A pulley is fixed at the top of a smooth inclined plane of inclination α to the horizon. A string passing over the pulley in a

principal plane is attached to a mass M on the inclined plane, and the other end supports a mass m hanging freely. Shew that M will just reach the pulley if m is detached after it has fallen a distance equal to $\frac{(M+m)\sin a}{m(1+\sin a)}$ of the original distance of M from the top.

2. A ball is rolled from one corner of a racket court, 64 ft. by 32 ft.; it strikes one of the longer walls at its middle point, rebounds, and strikes one of the shorter walls also at its middle point; find the coefficient of restitution.

3. If P is a point in a vertical circle, and the times of sliding are equal down all those chords of the circle through P which are on the side of the vertical through P remote from the centre, prove that the chords are all equally rough.

4. If the units of mass and time be doubled, the unit of length being unaltered, prove that the measure of the kinetic energy of a body is also doubled.

5. The time of describing any portion PQ of the parabolic path of a projectile is proportional to the difference of the tangents of the angles which the linear tangents at P and Q make with the horizon.

6. A smooth wedge, whose angle is 45°, weighs 5 lbs. and rests on a smooth inclined plane of elevation 45°, so that the upper surface of the wedge is horizontal. A weight of 4 lbs. is placed on the upper surface of the wedge: find its acceleration, and the pressure of the weight on the wedge. Does the weight move relative to the wedge?

Paper VII.

1. Two heavy particles begin to slide at the same instant from the common vertex of two smooth inclined planes: prove that the line joining them moves parallel to itself.

2. A train of mass 250 tons, including the engine, is drawn up an incline of 3 in 500 at the rate of 40 miles an hour by an engine of 600 horse-power; find the resistance per ton due to friction etc.

3. If a shot of 5 lbs. is fired horizontally from a gun of 7 cwt. with a velocity of 1100 ft. per sec., find the velocity of recoil of the gun; and compare the kinetic energies of the gun and shot.

4. A particle is projected with a given velocity and strikes a given plane through the point of projection horizontally: find the elevation of projection.

5. The hose of a fire engine is inclined at an angle of 45° to the vertical and discharges 2 gallons per second. The jet strikes a vertical wall at right angles, at a height of 35 ft. above the nozzle; shew that the pressure on the wall is approximately equal to the weight of 29·5 lbs. if a gallon contains 10 lbs. of water.

6. An inelastic sphere is in equilibrium supported by two smooth inclined planes, each of inclination a $\left(a < \frac{\pi}{4}\right)$. Prove that, if projected up one of the planes with velocity v, it will, after oscillating, finally come to rest in its original position in a time $\frac{v}{g \sin^3 a}$.

PAPER VIII.

1. The length of a smooth inclined plane is 80 ft., and its inclination is 30°; mark out upon it a part, equal to the height, through which a body sliding down the plane will move whilst another body would descend freely through the height.

2. Two masses of 5 lbs. and 3 lbs. are connected by a light string passing over a table, $6\frac{1}{2}$ ft. wide, at right angles to its edges, the smaller mass starting at a point 4 ft. below the edge. Find the distance fallen through by the larger mass in 2 secs., supposing the smaller mass to pass on and off the table without loss of velocity.

3. Two balls A and B, perfectly elastic, are dropped at the same instant from two points in the same vertical line, at heights of h_1 and h_2 feet above a horizontal plane: find the point where B, after rebounding from the horizontal plane, will meet A. $(h_1 > h_2.)$

4. If the unit of space be 2 ft., what must be the unit of time in order that one pound-weight may be the unit of force, and one pound the unit of mass?

5. Find the time which elapses between the directions of a projectile in the ascent and in the descent having the same inclination θ to the horizon, u and a being the initial velocity and elevation respectively.

6. Three equal particles move with velocities u_1, u_2, u_3 along three lines in one plane equally inclined to one another. Shew that the velocity of the centre of mass is

$$\tfrac{1}{3}\left(u_1^2 + u_2^2 + u_3^2 - u_1u_2 - u_2u_3 - u_3u_1\right)^{\frac{1}{2}}.$$

PAPER IX.

1. A mass of 200 lbs. falls from a height of 4 feet and drives an inelastic pile half an inch into the ground: find the mean pressure on the mass.

2. A bullet of mass 2 ozs. is fired into a block of wood of 25 lbs. mass, free to move in the direction of impact; and the block is observed to move off with a velocity of 8 ft. per sec. Find the original velocity of the bullet.

3. A string passing round a smooth fixed pulley is attached at one end to a weight, and at the other end to a smooth weightless pulley.

A string passes round this weightless pulley, and is attached at one end to a weight equal in mass to the first mentioned weight, and at the other end to the ground. All the hanging portions of the string are vertical. Prove that the first mentioned weight ascends with an acceleration equal to one-fifth of that due to gravity.

4. A particle is projected from a point on an inclined plane of elevation 45°, in a direction lying in the same vertical plane as the line of greatest slope, and making an angle of 15° with it; shew that the direction of motion when the particle meets the plane again will make an angle of 30° with the line of greatest slope.

5. If two unequal spheres moving with equal velocities (v) impinge directly, prove that the resulting loss of kinetic energy is $(1-e^2)\mu v^2$, where μ is the harmonic mean between the masses of the spheres.

6. A clock gains 5 seconds a day; calculate to 5 places of decimals by what proportion of its length the pendulum (assumed to be a simple one) ought to be lengthened.

PAPER X.

1. If a body fall through a distance of a ft. at two different places, and if the time of falling at one place be t secs. less, and the velocity acquired v ft. per sec. greater than at the other, prove that $\sqrt{\frac{2a}{g'}}-\sqrt{\frac{2a}{g}}=t$, and $\sqrt{gg'}=\frac{v}{t}$, when g and g' are the accelerations of gravity at the two places respectively.

2. A rocket ascending vertically with an initial velocity of 100 ft. per sec., explodes when it reaches its highest point, and the interval between the sound reaching the place of starting, and a place $\frac{1}{4}$ mile distant, is one second; find the velocity of sound.

3. A ball whose elasticity is $\frac{1}{3}$ falls from a height of 50 ft. upon a horizontal plane, and rebounds continually until its velocity is destroyed: find the whole space described.

4. A ball is fired at an elevation $\tan^{-1}\frac{1}{10}$ towards a person on the same horizontal plane as the gun; if the ball and the sound of the discharge reach him at the same instant, find the range, the velocity of sound being 1120 ft. per second.

5. Two small rings, of masses m and m', are moving on a smooth circular wire which is fixed with its plane vertical. They are connected by a weightless inextensible string. Prove that, as long as the string remains tight, its tension is $\frac{2mm'g\tan a\cos\theta}{m+m'}$, where $2a$ is the angle which the string when tight subtends at the centre, and θ is the inclination of the string to the horizon.

6. Heavy particles slide down chords of a vertical circle from rest at the highest point. Find the locus of the foci of the parabolic paths they describe after leaving the chords.

PAPER XI.

1. Bodies are let fall down a number of smooth inclined planes having a common vertex; prove that the locus of the points at which they all have the same given velocity is a horizontal line. If the planes are all equally rough, coefficient of friction μ, prove that the locus is still a straight line.

2. Shew that an erg $=737\times 10^{-10}$ foot-pounds approximately;
having given 1 metre $=3{\cdot}281$ feet, 1 poundal $=13825$ dynes,
and $g=32{\cdot}2$ ft.-sec. units.

3. At equal intervals a, on a smooth horizontal plane, n equal balls are arranged in a straight line. A velocity v is communicated to the first in the direction of the second; shew that the nth will begin to move with a velocity $v\left(\frac{1+e}{2}\right)^{n-1}$, after a time

$$\frac{a}{v}\left(\frac{1+e}{1-e}\right)\left[\left(\frac{2}{1+e}\right)^{n-1}-1\right],$$

e being the coefficient of restitution.

4. A perfectly elastic particle is projected with a velocity v in a vertical plane through a line of greatest slope of an inclined plane of elevation a; if after striking the plane it rebounds vertically, prove that it will return to its point of projection on the plane in a time

$$\frac{6v}{g(1+8\sin^2 a)^{\frac{1}{2}}}.$$

5. A parabola has its axis vertical, and its vertex at its lowest point; prove that the time of descent of a particle down any smooth chord to the lowest point is equal to the time of falling vertically to a horizontal line which is at a distance below the vertex equal to the latus rectum.

6. If a body of mass m make n complete Simple Harmonic Oscillations per second, shew that its kinetic energy when at the centre exceeds its kinetic energy when at distance x from the centre by $2n^2\pi^2mx^2$.

ANSWERS TO EXAMPLES.

I. *a*. (PAGE 3.)

1. $58\frac{2}{3}$ ft. per sec. **2.** 30 miles per hour. **3.** 22 ft. per sec.

4. 5 minutes. **5.** $1536\frac{32}{63}$ ft. per sec. **6.** $\frac{22at}{15k}$ ft.

7. $\frac{5a}{k}$ ft. **8.** $\frac{11}{6300}$. **9.** $\frac{11}{129600}$ ft. per sec.

10. 14 miles an hour. **11.** 15 : 22. **12.** $13\frac{1}{5}$ ft. per sec.

13. 10 miles. **14.** 36 days. **15.** 1 : 12 : 720. **16.** $\frac{ust'}{s't}$.

17. $14\frac{2}{3}$. **18.** $\frac{4}{5}$ of a foot.

I. *b*. (PAGE 6.)

1. 21 miles, 46 miles, 4 hrs. 48 min. **2.** 6 miles per hour.

3. 8 miles per hour. **4.** 29 miles, 32° W. of N.

5. $13\frac{1}{2}$ miles from the starting point. **6.** 10·2 miles per hour.

7. 9·6 yards per second. **8.** 15·7 ft. per sec.

9. 51·3 ft. per sec. **10.** 45 miles per hour.

II. *a*. (PAGE 11.)

1. 8 ft. per sec.; 16 ft. **2.** -3 ft.-sec. units; 10 ft.

3. $16\sqrt{3} = 27{\cdot}713$ ft. per sec. approx. **4.** 40 ft.-sec. units.

5. 16 secs. **6.** 10 secs.

7. In 2 secs.; 10 cms. from starting point. **8.** 40 ft.-sec. units.

9. 8 ft.-sec. units; 5 ft. per sec. **10.** 120 ft. **11.** 440 ft.

12. 0; 125 ft.; 375 ft. from the starting point in a negative direction.

13. 6 ft.-sec. units; 2 ft. per sec. **14.** 68 ft. **15.** $2\frac{1}{2}$ secs.

17. 8 secs. **22.** 208 ft. **23.** 7200. **24.** 4·4 ft.-sec. units.

25. 64 ft. **26.** 16 : 3. **27.** f becomes $4f$.

28. 4 secs. after the first particle starts; 16 ft. from O.

29. Acceleration $= \dfrac{a-b}{m-n}$; initial velocity $= \dfrac{b(2m-1)-a(2n-1)}{2(m-n)}$.

30. $7\frac{1}{3}$ ft. per sec. in a direction opposite to the acceleration; $51\frac{1}{3}$ ft. per sec. in the same direction as the acceleration.

31. $\frac{5}{3}$ ft.-sec. units; $20\sqrt{122}$ (=221 approx.) ft. per sec.; 220 ft. per sec.

32. In 9 secs.; 63 ft. from the starting point measured in the direction of the motion of the ball moving uniformly.

33. It travels forwards for $2\frac{3}{8}$ secs. over a distance of $15\frac{27}{32}$ ft.; velocity of departure from $A = 10\frac{1}{2}$ ft. per sec.; velocity of return to $A = 19\frac{1}{2}$ ft. per sec. in the opposite direction.

II. *b*. (Page 20.)

1. 8·5 ft. per sec., 22·5 ft. per sec., in 56 secs.
2. 7·8 ft. per sec. in 50 secs.
3. 28 ft. per sec. in 35 secs.
4. 2 ft. per sec., 3·5 ft. per sec.
5. 6·6 ft. per sec., 4·9 ft. per sec.
6. 3·5 ft., 6·9 ft., 4·5 secs.
7. 15·3 ft., 35 ft. 2·9 secs.
8. 2 secs. 10 ft. from the start, in a direction opposite to that of initial motion.
9. The acceleration is uniform and equal to -4 ft.-sec. units. Initial velocity 22 ft. per sec. It comes to rest in $5\frac{1}{2}$ secs.

II. *c*. (Page 24.)

1. 42, 34, 26 ft. per sec. Retardation of 8 ft.-sec. units.
2. 8, 10, 16 ft. per sec. Acceleration, 4 ft.-sec. units.
3. 3, 2, 6 ft. from O. 21 ft., ·004001 ft., 7 ft. per sec., 4·001 ft. per sec., 4 ft. per sec.
4. $150-10t$ ft. per sec.
5. $a+2bt$.
6. Velocity $6t^2$, acceleration $12t$.

III. (Page 28.)

1. In 3 secs.
2. In 8 secs.; 1024 ft.
3. 1 sec. and 7 secs.; 112 ft.
4. 192 ft.
5. 104 ft. per sec.
8. 1600 ft.
9. 32·2 ft.-sec. units.
10. $t = 5$ secs.; 64 ft. per sec.
11. In $8\frac{8}{9}$ secs.
12. 144 ft.; in $\frac{3}{2}(2-\sqrt{2}) = \cdot 88$ secs. approx.
13. 192 ft.
14. 576 ft.
15. It has fallen for 7 secs. when it overtakes the other.
16. In 2 secs. or 8 secs.
17. 256 ft. per sec.; 1024 ft.
18. 116 ft.
19. 192 ft. below the point of projection.
21. 160 ft.

22. 96 ft. per sec.; 0. 23. $\sqrt{gh}$.

24. 70·56 ft. below the point of projection. 25. $\left(\frac{h}{gt}-\frac{t}{2}\right)$ secs.

26. In $\frac{2}{g}\sqrt{v^2+2gh}$ secs. from its start.

27. $x=32t^2$ where x ft. is the unit of space and t secs. the unit of time.

28. 6 secs. 29. 4 secs. 30. $5\frac{1}{2}$ secs.

31. 6 secs. 32. 3·16 secs., 101·2 ft. per sec.

IV. (Page 37.)

1. Acceleration = 2 ft.-sec. units; space = 16 ft.
2. Acceleration = $\frac{4}{3}$ ft.-sec. units; velocity = 11 ft. per sec.
3. Acceleration = 4 ft.-sec. units; space = 1000 ft.
4. 384 poundals = 12 lbs. wt. 5. 16 lbs.
6. 640 ft. per sec.; 1280 ft. 7. 4 secs. 8. 48 ft. per sec.; 720 ft.
9. 1 : 16. 10. 3 lbs. wt. 11. 48 ft. 12. $\frac{1}{4}$ lb. wt.
13. 45 grammes. 14. 6 secs. 15. 981 grammes.
16. $\frac{60\,pgt}{m}$ ft. per sec.; $\frac{1800\,pgt^2}{m}$ ft. 17. 10 secs. 18. $6\frac{4}{11}$ miles.
19. 800 lbs. wt. 20. $\frac{3}{50}$ ft.-sec. units.
21. $\frac{P-W}{W}g$ ft.-sec. units; $\frac{yP}{l}$, where l is the length of the chain and the assigned point is at a distance y from its lower end.

V. (Page 46.)

1. 8 miles an hour, in a direction 30° W. of N.
2. 26 ft. per sec., at an elevation of $\tan^{-1}\frac{12}{5}$ (=67° 23′).
3. 3 miles an hour. 4. $10\sqrt{5}$ (=22·36 approx.) miles.
5. $\sqrt{19}$ (=4·36 approx.) ft. per sec.
6. 10·9 ft. per sec. at an angle of 20° with the greater velocity.
7. 31° with the bank. 5·83 miles per hour.
8. 14·4 ft. 9. 33·17 yds. per minute.
10. 8 ft. per sec., and 4 ft. per sec. 11. 1·5 miles per hour.
12. Each component $=\frac{v\sqrt{3}}{3}=v\,.\,5{\cdot}8$ approx.

13. 50 ft. **14.** 12 miles. **15.** $4\sqrt{3}\,v$ ft. $=6\cdot928\,v$ approx.

16. $3\sqrt{7}\,(=7\cdot94)$ ft. per sec. making an angle $\sin^{-1}\left(\frac{\sqrt{21}}{14}\right)$ or $19^\circ\ 6'$ with the resultant velocity.

17. $\sqrt{31}\,(=5\cdot57)$ ft. per sec.

18. v from the fifth angular point to the centre.

19. 26 ft. **20.** $2\cdot2$ ft. per sec.

21. Up stream, at an angle of 45° with the bank.

23. Half an hour. **24.** $2\sqrt{2}\,u=2\cdot83\,u$, south-west.

26. 5 hours 20 minutes.

VI. (Page 55.)

1. 72 ft.; 48 ft. per sec. **2.** 45°. **3.** 60°.

4. $1716\cdot75$ centimetres. **5.** 75 secs. **6.** 108 yds.

7. $2(\sqrt{2}-1)=\cdot828$ secs. approx. **8.** $\frac{l^2-h^2}{l}\sqrt{\frac{g}{2h}}$. **17.** 45°.

18. 96 ft. per sec.; 144 ft. **19.** $1\frac{3}{5}$ lbs. wt. **20.** $5\frac{13}{220}$ miles.

21. 11 tons wt. **22.** Half a ton wt. in each case.

23. $54\frac{6}{11}$ miles per hour. **24.** $1\frac{3}{8}$ miles. **25,** u.

26. $\frac{P\cos a - mg\sin a}{m}$. **27.** 12 ft. per sec. **28.** 36 ft.

29. $3\cdot43$ ft.-sec. units. **30.** 4 tons wt.

31. The particle sliding down BC arrives at C first, and is then 6 inches from the other.

32. $2\cdot7$ lbs. wt. **33.** 2 lbs. wt.

34. 4 secs. after the first started, 128 ft. from the top.

35. 22°. **36.** $3\frac{1}{2}$ lbs. wt.

39. 32, 96, 160 ft. 3 secs. **40.** 30°.

VII. (Page 65.)

1. 8 ft.-sec. units; $3\frac{3}{4}$ lbs. wt.; 16 ft. **2.** $4\frac{4}{9}$ lbs.

3. 24 ft. per sec. **4.** $15\frac{3}{4}$ ozs. wt. **5.** By 3 lbs. wt.

6. $\frac{1}{2}$ sec. **7.** $\frac{v}{g}$ secs. **8.** 16 ft. and 48 ft.

9. $1\frac{1}{4}$ secs. **10.** $1\frac{1}{3}$ lbs. wt. **11.** 3 secs.

12. 8 ft.-sec. units; weight of $\frac{3m}{4}$ lbs. **13.** 12 lbs.

14. $\sin^{-1}\left(\frac{5}{8}\right) = 38° \, 41'$. **15.** $7\frac{7}{8}$ lbs. wt. $5\frac{5}{8}$ lbs. wt.

16. 128 ft. **17.** 4 lbs. wt. $5\frac{3}{8}$ lbs. wt.

18. 5 : 3. **20.** 2 secs. **21.** 4 ft.-sec. units; $1\frac{7}{8}$ lbs. wt.

22. $16(\sqrt{3}-1)$ (= 11·713 approx.) ft. **23.** $1\frac{1}{2}$ secs. **24.** 3 secs.

28. Acceleration $= \frac{3g}{403}$; $g = 981$ cm.-sec. units approx.

31. If m_1 be the mass on the inclined and m_2 the mass on the horizontal plane, acceleration $= \frac{m_1 g \sin a}{m_1 + m_2}$ and tension $= \frac{m_1 m_2 g \sin a}{m_1 + m_2}$, where a is the inclination of the plane.

32. $\left(\frac{W_2 - W_1 \sin a}{W_2 + W_1}\right) g$. **33.** $g = 32$ ft.-sec. units. **34.** 80 ft.

36. Initial velocity = 400 cms. per sec.; ratio of masses = 277 : 213; acceleration = 128 cm.-sec. units.

38. $2\frac{1}{2}$ lbs. wt. and $3\frac{1}{3}$ lbs. wt. **39.** $7\frac{7}{8}$ lbs. wt. and $4\frac{3}{8}$ lbs. wt.

VIII. *a*. (Page 71.)

1. 70 lbs. wt.; 56 lbs. wt. **2.** 4 ft.-sec. units.

3. The wt. of m lbs., in a vertical direction. **4.** 45 lbs. wt.

5. Zero. **6.** 3 lbs. wt. **7.** 196 lbs. wt.

8. 810 lbs. wt. **10.** 15 lbs. wt. **11.** $2\frac{2}{3}$ ozs. wt.

VIII. *b*. (Page 74.)

1. $5\frac{1}{3}$ ft. **2.** 27 ft. **3.** $38\frac{1}{28}$ tons wt.

4. 16 ft. per sec. **5.** $25\frac{25}{32}$ tons wt.

7. $\frac{180\sqrt{7}}{7} = 68$ miles an hour nearly. **8.** $\frac{3}{28}$ ft.-sec. units.

10. ·48 ft.-sec. units. **11.** $\mu = \frac{1}{2}$. **12.** $\frac{2gh - v^2}{2gh} \tan a$.

13. 80 ft. per sec. **14.** $\frac{\sqrt{3}-1}{3} g = 7·8$ ft.-sec. units approx.

15. 79 poundals $= 2\frac{15}{32}$ lbs. wt. **16.** 4 secs. **17.** 1 : 160.

18. $\frac{l'}{l} = \frac{T'}{T} = \frac{\sin a - \mu \cos a}{\sin a + \mu \cos a}$. **19.** 170 : 13.

Revision Questions. IX. *a*. (Page 77.)

13. (1) Zero, (2) no difference.

14. 3 ft. per sec. in the direction of the greatest velocity.

15. 3 ft. per sec. bisecting the angle of 120°.

16. 120, 7200, $\frac{2}{3}$, 4. 17. 180, 10800.

18. 30, $\frac{1}{2}$, $\frac{1}{120}$, $\frac{5280}{120}=44$. 19. 1920, 115200, 115200.

20. 8, 16, 80, $8t$ ft. per sec. 21. 8, 14, 20, $2+6t$ ft. per sec.

Revision Paper. IX. *b*. (Page 79.)

1. $\frac{2a}{15b}$. 2. 9. 3. 1 sec. 4. 26 lbs. wt.

5. $10\frac{2}{3}$ ft.-sec. units, $5\frac{1}{3}$ lbs. wt., 32 ft. per sec. 16 ft.

6. The body has a retardation of 2 ft.-sec. units, and comes to rest in $5\frac{1}{2}$ secs.

Revision Paper. IX. *c*. (Page 80.)

1. 1528. 2. 3·16 secs., 101·2 ft. per sec.

3. 159·4 lbs. wt., 140·6 lbs. wt. 4. 3 ft.-sec. units, 6 ft. per sec.

5. $\sin^{-1}(\cdot 9219)=67° 12'$. 6. $2P=mf$.

Revision Paper. IX. *d*. (Page 80.)

1. $3\frac{1}{3}$ ft. per sec. 2. In 4 secs., 160 ft. from B.

3. 32 ft. 4. 6 miles an hour.

5. 64 ft. 6. $8\frac{8}{15}$ oz. wt.

Revision Paper. IX. *e*. (Page 81.)

1. 45. 2. 10·5 secs. 3. $22\frac{11}{12}=22\cdot 92$ lbs. wt.

4. (1) It just carries the load.
   (2) It would break.
   (3) It could carry a greater load.

5. 5·3 ft. per sec., 6·1 ft. 6. 32·2.

Miscellaneous Examples. IX. *f*. (Page 82.)

1. 30 ft. per sec. 2. 96 ft.

3. In $1\frac{2}{3}$ secs. after the second is projected, at a height of $168\frac{2}{3}$ ft.

4. 1600 ft. 5. 5 : 13. 6. 60°. 7. 360 ft.

8. $3\frac{7}{16}$ minutes. **9.** $\frac{36}{175}$ ft.-sec. units. **10.** $3\frac{9}{70}$ secs.

11. $1\frac{7}{48}$ tons wt. **12.** $33\frac{1}{18}$ tons wt. **13.** $1+2\sqrt{2}:3$.

14. $m\left[\frac{gh}{l}+\frac{2l}{t^2}\right]$ poundals. **15.** $m=5$ ozs. mass. **16.** $57\frac{1}{2}$ ft.

17. At an angle of 30° with BA. **18.** 26 ft.

20. Its increase of velocity is 7 ft. per sec. in each second.

21. 60 miles an hour; 10 hours, 25 minutes. **22.** 99 ft.

24. $2\frac{8}{11}$. **25.** 10. **27.** Wt. of $\frac{2}{3}m$ lbs.; wt. of $\frac{1}{3}m$ lbs.

28. (1) $\frac{4}{7}$ tons wt.; (2) $20\frac{4}{7}$ tons wt. **31.** $34\frac{2}{3}$.

33. If m lbs. be the mass of unit length of the string, and y the distance of a point from the descending end, the tension at that point $=myg\left(1-\frac{x}{l}\right)$ poundals.

34. $21\frac{9}{11}$.

35. In $1\frac{1}{4}$ secs.; 37 ft. from the point where the lower particle started.

39. $\sqrt{\frac{gl}{2}}$. **40.** Time $=91\frac{2}{3}$ secs.; distance $=2688\frac{8}{9}$ ft.

41. With a velocity of 20 ft. per sec.; at an angle $\tan^{-1}\left(\frac{3}{4}\right)$ with the direction of the ship's motion.

42. $10\frac{10}{11}$ miles. **43.** $1058\frac{3}{4}$ ft. **44.** 120°, 135°, 105°.

45. 48·4 ft. per sec. approx.; 2·06 secs. approx. **46.** 3220.

47. 15 poundals will just prevent the body from sliding down.

48. $3\sqrt{3}$ miles per hour. **49.** 27 poundals; $13\frac{1}{2}$.

50. $2\frac{3}{4}$ secs.; 64 ft. per sec. **55.** $\sqrt{19}u$. **56.** 2 : 1.

60. $\frac{4h}{3}$ ft.

61. 3 secs. after the second body starts, at a height of 240 ft.

62. 196 ft. **63.** 1520 ft. **64.** 6 lbs. wt.

65. $\frac{5\pi a}{v}$. **66.** A horizontal straight line.

68. Their distance apart at time $t=\sqrt{a^2-2aut+2u^2t^2}$, where a is the distance of the station from the junction, and u is the velocity of each train.

72. He must aim at a point distant $\frac{uv\left[\sqrt{v^2-u^2\sin^2\alpha}-u\cos\alpha\right]}{v^2-u^2}$ ahead of the apparent position of the object, where $v=$ distance of the man from this apparent position, and $\alpha=$ the angle between this distance and the line of flight of the object.

73. $\dfrac{Ps}{(\mu Q - P)}$. 74. $8\sqrt{\dfrac{165a\,(f+f')}{ff'}}$ secs.

76. $P = 266\frac{2}{3}$ lbs. wt. 77. 300 ft. 78. $\dfrac{\sqrt{3}}{6}$.

80. $\left(\dfrac{h}{t} + \sqrt{2gh}\right)$ ft. per sec. 81. 11 : 5400.

84. Tension of string $= \dfrac{2W}{3}$. 86. 25·6 ft.

87. $\dfrac{2m_1m_2g}{m_1+m_2}$ poundals; $\dfrac{2M_1M_2(m_1+m_2)+4m_1m_2(M_1+M_2)}{(m_1+m_2)(M_1+M_2)}\,g$ poundals.

X. *a*. (PAGE 96.)

1. 12320 ft.-lbs. 2. 11·4 secs. 3. $5\frac{1}{11}$ H.P.
4. $1\frac{1}{3}$ H.P. 5. 1 min. $18\frac{2}{3}$ secs. 6. 50 minutes.
7. 2475 ft.-lbs., 15 secs. 8. $40\frac{8}{11} = 40·73$. 9. $41\frac{1}{11} = 41·09$.
10. 0·176. 11. 2240. 12. 110, 275.
13. 22·4.
14. 147840 ft.-lbs.; 2·24 H. P. 15. 6 ft.-tons.
16. 26,400 cub. ft. 17. $4\frac{13}{28}$. 18. 2240 ft.-lbs.
19. 303,750 ft.-lbs. 20. $5\frac{1}{11}$. 21. $1\frac{11}{14}$.
22. $21\frac{1}{3}$. 23. 3 miles, 970 yards. 24. $25\frac{3}{8}$.
25. $5866\frac{2}{3}$ lbs. wt. 27. 660,000 ft.-lbs.; 30 H.P.
28. $\dfrac{mv}{1100}\left(\sin\alpha + \mu\cos\alpha + \dfrac{v}{gt}\right)$. 29. $750\frac{14}{15}$.
30. $50(2\sqrt{3}-3)\,[=23·2]$ ft.-lbs. 31. 295·68.
32. $7\frac{1}{3}$ ft. 33. $13\frac{1}{5}$ hours.

X. *b*. (PAGE 100.)

1. 27 ft.-lbs. 2. 500 ft.-lbs.
3. 250 ft.-lbs. 4. 60,000 ft.-lbs.

X. *c*. (PAGE 106.)

1. 45,000, 22,500 ft. 2. $2\frac{7}{9}$.
3. $\dfrac{P}{2g}(v^2 \sim v'^2)$. 4. 1920 tons wt. 5. 12·5 nearly.
6. 40,960 units of energy. 7. 307,200 units of energy.
8. $109\frac{3}{8}$. 10. 3·125 lbs. wt.
11. 55561 ft.-lbs. approx. 12. 44·2 approx. 13. $1\frac{5}{6}$ miles.

14. 12·4. **15.** 25 : 4. **16.** $91\frac{2}{3}$.

17. 10·85 ft. approx. **18.** 32000. 25 lbs. wt.

19. 140 ft.-lbs. 10 ft.-lbs.

20. 480 ft.-poundals. Decrease of 480 ft.-poundals.

21. $4\frac{1}{16}$ tons wt. **22.** $37\frac{29}{33}$. **23.** 28 : 165.

24. 1,815,000,000 ft.-lbs. **25.** $4\frac{31}{336}$. **26.** 60,000 ft.-lbs.

27. 1,108,800 ft.-lbs. **28.** 24 ft.-poundals. $3\frac{1}{2}$ lbs. wt.

29. 168. **30.** 40 ft. per sec.

32. (1) 1250 lbs. wt. (2) 3600 lbs. wt. **33.** $2149\frac{5}{7}$ ft. tons.

36. mgh. **37.** 37. **38.** 102 tons wt.

39. 5625 lbs. wt. **40.** m lbs. wt. **41.** Rh.

XI. *a.* (PAGE 116.)

1. 24 miles an hour, in a direction 30° W. of N.

2. 6 miles an hour, in a direction due north.

3. $6\sqrt{5}$ ($=13·42$) miles an hour, in a direction $\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ ($=26° 34'$) N. of E.

4. 60 miles per hour, and 30 miles per hour.

5. It meets him at an angle $\tan^{-1}\frac{1}{2}$ ($=26° 34'$) with the horizon and with a velocity of $44\sqrt{5}$ ($=98·4$) ft. per second.

6. It blows 5 miles an hour, at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ ($=53° 8'$) N. of W.

7. 4 miles; 36 minutes. **8.** 60 miles an hour.

9. 10 miles an hour from the N.W.

10. $13\frac{7}{11}$ and $27\frac{3}{11}$ miles per hour.

11. $3v$, greatest when at the same point; v, least when at opposite ends of a diameter.

12. The rain comes from behind the man at an angle of 45° with the horizon, and with a velocity of $4\sqrt{2}$ miles per hour.

13. 65 miles an hour. **14.** Their velocities are equal.

15. At an angle $\tan^{-1}\frac{3}{4}$ ($=36° 52'$) W. of N. **16.** 6 miles.

17. $6\sqrt{2}$ miles per hour; S.W. **18.** At an angle $\tan^{-1} 2$ ($=63° 26'$).

20. At 12.45, when they are $\frac{45\sqrt{2}}{4}$ ($=15·9$) miles apart.

22. $3\frac{27}{31}$ lbs. wt. and $7\frac{23}{31}$ lbs. wt.

23. $5\frac{1}{3}$ lbs.

24. $15\frac{15}{31}$ ft.-sec. units.

25. $\frac{g}{3}$, $\frac{2mg}{3}$ poundals.

26. $\frac{3g}{13} = 7 \cdot 4$ ft.-sec. units approx., $4\frac{8}{13}$ lbs. wt.

27. $\frac{5m}{17}$ lbs. wt. $\frac{3g}{17} = 5 \cdot 65$ ft.-sec. units approx.

28. $5\frac{5}{8}$ lbs. wt. 2 ft.-sec. units.

29. (1) $\frac{3m_1 m_2}{m_1 + 4m_2}$ lbs. wt. $\frac{m_1 - 2m_2}{m_1 + 4m_2} g$ downwards.

(2) $\frac{g}{2}$ upwards. (3) $2g$ upwards.

30. 4·8 oz. wt.

XI. *b.* (Page 120).

2. 7·5 miles per hour.

3. 8·3 miles per hour, 67° N. of E.

4. 22° 54′ N. of E.

5. 80° E. of S., 9° 13′ N. of E.

XI. *c.* (Page 122.)

2. $\sqrt{\frac{2053}{1040}}$ secs.

3. $1\frac{1}{2}$ lbs. wt.

5. 3·2 feet. 38·4 feet.

7. $g \sin \alpha$.

XI. *d.* (Page 125.)

1. 1·17 radians per sec., approx.

2. 5 ft. per sec.

3. 5·24 radians per sec., approx.

4. 4; $114\frac{6}{11}$°.

5. 1 : 12 : 720.

6. $\frac{D-d}{D} v$.

7. 22 radians per sec.; 44 ft. per sec.

9. $\frac{2\pi a}{u}$.

10. $\frac{\pi}{43200}$ radians per sec.

11. 10 inches.

12. $\frac{1}{24}$.

15. $45\sqrt{2}$ $(=63 \cdot 64)$ miles per hour, at an angle of 45° with the horizon.

16. 90 miles an hour; 0.

18. 10; 10.

20. $\frac{1}{2}$, $\frac{\pi}{12}$.

22. 22 radians per sec.

23. $\frac{11\pi}{6}$ radians per hour $= \frac{11\pi}{21600}$ radians per sec.

XI. *e*. (Page 128.)

1. AC represents its velocity after one second.
2. Produce BC to D making $BD=3BC$. AD represents the velocity required.
3. 13·68 ft. per sec., 4·56 ft.-sec. units.
4. Accel. 4·8 ft.-sec. units, velocity 18·3 ft. per sec., making 18° with the initial direction.
5. $AB=BC=CD$ and so on, and $ABCD$ is a str. line.
6. 34·6, 33, 36, 42, 50·4 ft. per sec.
   17°, 37°, 56°, 72°, 82° with North.
8. 15 ft. per sec. bisecting angle BAC.
9. 8 ft. per sec. at right angles to AB.

XII. (Page 138.)

1. $1\frac{1}{2}$ lbs. wt. **2.** 16 ft.-sec. units. **3.** $\frac{11}{45}$ tons wt.

4. $\frac{33}{80}$ lbs. wt. **5.** $38\frac{2}{11}$ times. **6.** $30\frac{1}{4}$ lbs. wt.

7. $\tan^{-1}\left(\frac{11}{160}\right)$. **8.** $\tan^{-1}\left(\frac{v^2}{rg}\right)$.

9. 3 in. approx. **12.** 38.

13. $4\sqrt{10}\,(=12·65)$ stone at an angle $\tan^{-1}\frac{1}{3}$ with the vertical.

14. $\frac{10\sqrt{3}}{3}\,(=5·77)$ lbs. wt. at an angle of 30° with the vertical.

15. 6 lbs. wt. at an angle $\cos^{-1}\left(\frac{2}{3}\right)$ with the vertical.

16. $\sqrt{2gc}$. **18.** ·10125. **19.** $m_1v_1^2 : m_2v_2^2$.

20. At a perpendicular distance $\frac{g\cot a}{\omega^2}$ from the axis. **21.** $15\frac{3}{11}$.

22. $8\sqrt{3}\,(=13·86)$ lbs. wt.; $\frac{\pi}{4}\sqrt[4]{300}\,(=3·3)$ secs.

23. $\frac{165}{224}$ oz. wt. **26.** $\frac{3}{4}m$ must be removed.

27. $29m$ poundals. **30.** 1·71 oz. wt. approx.

31. $m\pi^2$ and $\frac{3}{4}m\pi^2$. **32.** 2 secs. **35.** $6\frac{1}{4}$ lbs. wt.

36. 84 tons wt. **37.** $1\frac{1}{8}$ lbs. wt.

XIII. *a*. (Page 149.)

1. $\sqrt{31296} = 176 \cdot 9$ ft. per sec., approx.; 156·25 ft. 2. 156·25 ft.
3. 192 ft. per sec. 4. 15° or 75°. 5. 96 yds.
6. 400 yds. 7. 45°. 8. 60°. 9. 448 ft. per sec.
10. $\tan^{-1}\left(\frac{4a}{b}\right)$. 14. At an elevation of 60°. 18. 98 yds.
19. (1) 11 secs. at a ht. of 1936 ft.
(2) $8\frac{1}{4}$ secs. and $13\frac{3}{4}$ secs. at a ht. of 1815 ft.
20. Horizontal; 64 ft. 21. 150 ft. 22. 15° or 75°.
23. $16g$ ft.; 45° with the horizon. 24. 15° or 75°.
25. 225 ft.; 125 ft. per sec. 26. $\tan^{-1}\left(\frac{2k}{h}\right)$, $\sqrt{\frac{g(4k^2+h^2)}{2k}}$.
27. $40\sqrt{2}\,(=56 \cdot 57)$ ft. per sec. 28. 50 ft. per sec.
29. $4\frac{1}{4}$ secs. 30. $80\sqrt{3}\,(=138 \cdot 56)$ ft. from the foot of the tower.
31. $128\sqrt{2}\,(=181)$ ft. per sec. 32. 144 ft.
36. $32\sqrt{3}\,(=55 \cdot 43)$ ft. from the foot of the tower with a velocity of $32\sqrt{3}\,(=55 \cdot 43)$ ft. per sec.
37. With a velocity of $40\sqrt{2}\,(=56 \cdot 57)$ ft. per sec. at an elevation of 45°.
38. At an elevation of 15° or 75°.

XIII. *b*. (Page 155.)

1. In 3 secs. $3g$ ft. horizontally, $\frac{9g}{2}$ ft. vertically from the starting point of the first particle.
3. In $2\frac{1}{2}$ secs. 100 ft. vertically below the middle point of AB.
4. 44 ft.
7. 45°, 63°, 72°, 76° with the horizon.

XIV. (Page 169.)

1. $\sqrt{3}\,(=1 \cdot 73)$ secs.; 48 ft. 2. $\frac{u^2}{24}$. 3. $\frac{u}{16}$; $\frac{u^2\sqrt{2}}{16}$.
4. At time $\frac{u \sin(\alpha-\beta)}{g \cos\beta}$; when the distance from the plane $=\frac{u^2 \sin^2(\alpha-\beta)}{2g \cos\beta}$.

5. $533\frac{1}{3}$ ft. 6. $62\frac{1}{2}$ ft.; $\frac{5\sqrt{2}}{8}$ $(=\cdot 88)$ secs.

8. $d-\sqrt{d^2-h^2}$. 12. $\frac{2u}{g \sin \alpha}$. 13. 30°.

23. $\frac{ubt}{\sqrt{4a^2+b^2}}-b$ ft. 25. $\tan^{-1}\frac{1}{10}$ $(=5^\circ\ 43')$ approx.

29. A circle whose centre is at the point of projection.

31. $y+x \cot 2\alpha=0$ where α is the angle of projection; $\frac{u}{g \sin \alpha}$ where u is the velocity of projection.

35. $30\frac{1}{4}$ ft.; $\cdot 06$ inch approx.

37. The angular distance of the required point from the highest point on the wheel $=\cos^{-1}\left(\frac{gr}{u^2}\right)$, where r is the radius of the wheel and u the tangential velocity of the point.

Revision Paper. XV. *a*. (Page 174.)

1. 15 and 30 miles an hour. 2. 15 ft.

3. $\left(mh+\frac{mv^2}{2g}\right)$ ft.-lbs. 4. $\frac{vp}{r^2}$.

5. 25 ft. per sec.

6. Velocities 108, 95, 92 ft. per sec. approx.
Angles 57°, 74°, 93° with the vertical.

Revision Paper. XV. *b*. (Page 174.)

1. 18 inches per hour. 2. 4 ft. 18·63 ft. approx.

3. 11·574 tons wt. 28,233,000 ft.-lbs. 4. 40 ft. per sec.

5. ·0089 ft.-sec. units, 47281×10^{16} lbs. wt.

Revision Paper. XV. *c*. (Page 175.)

1. $\sin^{-1}\cdot 8=53^\circ\ 8'$ with AB. 2. 3000 ft.-poundals. $93\frac{3}{4}$ lbs. wt.

3. $\frac{3g}{5}$, $\frac{4mg}{5}$.

4. 19·1 ft. per sec. making angle 125° with its first direction.
Acceleration 6·4 ft.-sec. units nearly.

6. $3\frac{1}{17}$ lbs. wt.

Revision Paper. XV. *d.* (Page 176.)

1. $\tan^{-1}\frac{2}{11} = 10^\circ\, 18'$ with the direction of the train.
2. 5·8 secs. after the first has started.
3. 108,000 ft.-lbs. per c. ft., 17280 ft.-lbs. per gallon. 960,000 ft.-lbs. for a penny.
4. It has an acceleration of 3·2 ft.-sec. units downwards.
5. v, 0.

Revision Paper. XV. *e.* (Page 176.)

1. At an angle of $142\frac{1}{2}^\circ$ with the direction of the struck ship.
2. 205 ft. 100 and 64 ft. per sec.
3. $2\frac{1}{16}$ pence. **4.** 9 : 7. **5.** 30 miles an hour.

Revision Paper. XV. *f.* (Page 177.)

1. $\tan^{-1}\frac{4}{11} = 19^\circ\, 59'$ with the horizon. **3.** 10 lbs. wt., $1\frac{1}{4}$ lbs. wt. increase.
4. Velocity $= 4t - 3$ ft. per sec. Acceleration $= 4$ ft.-sec. units.
5. 55°. **6.** $\sqrt{\frac{\mu g}{r}}$.

Revision Paper. XV. *g.* (Page 178.)

1. 48 ft., 36 ft. **2.** $8\frac{1}{4}$ lbs. wt.
3. $2\frac{2}{3}$. **5.** $\frac{2}{5}$ oz. wt. $4\sqrt{2} = 5{\cdot}66$ ft. per sec.
6. The particle projected along AT' arrives first.

Revision Paper. XV. *h.* (Page 179.)

1. 5 ft. per sec. **3.** $33W$ ft.-lbs. of work done.
4. $\frac{11-\sqrt{2}}{3} = 3{\cdot}195$ ft.-sec. units. 3195 poundals = 32 lbs. wt. approx.
5. 19·7 lbs. wt. approx.

XVI. *a*. (PAGE 182.)

1. 100 units of impulse; 10,000 poundals. 2. 15 lbs.

3. 360 units of impulse.

4. $\frac{3mu}{2}$ units of impulse; $\frac{3mnu}{2}$ poundals.

5. 5 units of impulse. 6. 10 ft. per sec. 7. 3 : 4.

8. $\frac{mgt}{2}(2+\sqrt{2})$ units of impulse. 9. 4 : 3. 10. 3 : 8.

11. $14\frac{2}{7}$ ft. per sec. 12. $43\frac{121}{128}$ tons wt. 13. $14\frac{1}{16}$ ft.

14. 537,600 units of impulse; 302,400 lbs. wt.; 6 inches.

15. 2400 ft. per sec. 16. $10\frac{16}{45}$ ft. per sec.; $5\frac{1225}{2744}$ ft.

XVI. *b*. (PAGE 190.)

1. They are inelastic. 2. The effect is a question of momentum.

3. $11\frac{1}{4}$ units of impulse or momentum. 4. e^2h, e^4h.

5. Along the line of impact. 7. $u\sin\alpha$ along the plane.

8. $2mu$ where m is the mass of the ball.

9. $2mu\cos\alpha$ where m is the mass of the ball.

10. It will move on in its original direction.

11. $u\cos(\pi - ABC)$ along BC.

12. If a right hexagonal cylinder were placed on a smooth table, and a perfectly elastic ball were rolled from the middle point of one side so as to strike the middle point of the next side, it would describe a regular hexagon.

13. None. The vertical effect on the velocity of the ball at the impact is nil.

14. He might take off an article of clothing and throw it away.

15. $5\frac{1}{3}$ ft. per sec. and $6\frac{1}{3}$ ft. per sec.

16. $4\frac{4}{5}$ and $8\frac{4}{5}$ ft. per sec. respectively in the direction of motion of the larger ball.

17. $e=\frac{2}{3}$. 21. u. 24. u. 25. $\left(\frac{2}{3}\right)^{n-1}u$.

XVI. *c*. (PAGE 196.)

1. The first ball moves at right angles to the line of impact with velocity $\frac{u}{2}$, and the other along the line of centres with velocity $\frac{u\sqrt{3}}{2}$.

4. They will move with velocities $\frac{\sqrt{7}\,.\,u}{3}$ and $\frac{\sqrt{13}\,.\,u}{3}$, at angles $\tan^{-1} - 3\sqrt{3}$ and $\tan^{-1}\frac{3\sqrt{3}}{5}$ with the line of impact.

5. They will move parallel to one another at an angle of 45° with the line of centres, with velocities $\frac{u\sqrt{2}}{2}$ and $\frac{u\sqrt{6}}{2}$ respectively.

7. $e:1$.

8. (1) They will each move after impact with velocity $u\sqrt{\sin^2 a + e^2\cos^2 a}$, at an angle $\tan^{-1}\left(\frac{\tan a}{e}\right)$ with the line of centres.
(2) $a = \tan^{-1}(\sqrt{e})$.

9. $v\frac{\sqrt{43}}{8}$ ft. per sec. at an angle $\tan^{-1}\left(\frac{4}{3\sqrt{3}}\right)$ with the line of impact.

10. $2mu\sin\frac{a}{2}$ making an angle $\frac{\pi + a}{2}$ with the original direction of the moving mass.

XVI. *d.* (Page 198.)

1. $5\frac{5}{7}$ ft. per sec. 2. 16 ft. per sec.
3. $5\frac{1}{3}$ ft. per sec. $37\frac{1}{3}$ units of momentum. 4. 7 ft. per sec.
5. 16 ft. per sec. 6. 6 inches, 48 units of momentum.

XVI. *e.* (Page 203.)

1. 16 ft. per sec. 3. 2250 ft. per sec. 6. 4 ft. per sec.
9. The portions are equal.
10. At a point distant $\frac{1}{18}$th of the circumference from the start.
12. 600 lbs. wt. 14. 2 ft. per sec.
15. The smaller ball will move after impact with a velocity of $\frac{20\sqrt{41}}{9}$ $(=14{\cdot}23)$ miles per hour at an angle $\tan^{-1} 9$ $(=83^\circ\,40')$ with the line of impact, and the larger will move with a velocity of $\frac{40\sqrt{2}}{9}$ $(=6{\cdot}29)$ miles per hour along the line of impact.
16. 4 ft. per sec.; $1\frac{1}{2}$ ft.
22. A parabola in a vertical plane at right angles to the line of rails and having its axis vertical.
23. $666\frac{2}{3}$ yds. 24. 3 ft.

28. Horizontally or at an elevation $\tan^{-1} 2$.

37. $eu^2 \sin 2\alpha = gk$. **38.** $k\left(\frac{1}{e^2} - 1\right)$.

41. From a point in either side cushion distant 5 ft. from the top cushion.

XVI. *f*. (Page 212.)

1. 12 ft. **2.** $4\sqrt{3}\ (=6{\cdot}93)$ ft. per sec. $1\frac{1}{3}$ ft.

4. $\frac{mu^2}{4}$. **5.** $274{\cdot}4$ cwt. $=13{\cdot}72$ tons wt.

6. $16\frac{2}{3}$. **7.** $\frac{3mu^2}{16}(1-e^2)$. **10.** 1980 ft. per sec.

11. $\frac{2mv}{m'}$. $\frac{2mv^2}{m'}(2m'-m)$. **12.** 14,400 tons wt.

13. $\frac{mv}{M+m}$. He must throw the ball in the direction of his motion with a velocity $\frac{(2M+m)\,v}{M+m}$.

14. By throwing the ball away in the direction of his motion with velocity v.

15. $606{\cdot}76$. **16.** 1350 ft. per sec.

17. $5\frac{65}{87}$ ft. per sec. in each case.

19. $\sqrt{\frac{6b}{g}}$ secs.

20. $\sqrt{\frac{m-m'}{m+m'}gh}$, $\frac{\sqrt{(m^2-m'^2)\,gh}}{2m}$. **21.** $\frac{v}{2}$.

22. Velocity of ball $= \frac{1+3e}{2} v$ upwards.

Velocity of each pan $= \frac{1-e}{2} v$.

XVII. (Page 221.)

1. 9 lbs. wt.; 16 ft. per sec. **2.** 36 lbs. wt.

6. 3 lbs. wt.; when it is at an angular distance $\cos^{-1}\left(\frac{1}{3}\right)$ from the lowest point.

8. It will rise to a point at an angular distance $\cos^{-1}\left(\frac{5\sqrt{2}-3}{7}\right)$ from the lowest point.

10. 5 : 23 : 14.   11. 16 ft. per sec.

12. When the radius to the point makes an angle θ with the horizon, acceleration along the arc $= g\cos\theta$; acceleration at right angles to the arc $= 2g\sin\theta$; velocity $= \sqrt{2gr\sin\theta}$; pressure on arc $= 3mg\sin\theta$ poundals, where m lbs. is the mass of the particle.

14. 16128 units of momentum; wt. of 9 cwt.; wt. of $4\frac{1}{2}$ cwt.

15. $\sqrt{gr}\sin\theta$, where r = radius of wire, and θ the angular distance the bead has moved through from the highest point.

16. $\frac{16r}{27}$.

17. Angular distance from the highest point $= \cos^{-1}\left(\frac{u^2+2gr}{3gr}\right)$.

19. $\tan^{-1}\left(\frac{5\sqrt{5}}{8}\right)$.

XVIII. (Page 234.)

1. $2\frac{6}{7}$ ft. per sec.   2. $\sqrt{13}$ ft.   4. 4 secs.

5. 81 ft. nearly.   6. $15\frac{5}{7}$ secs.   7. 54982 nearly.

8. 840.   9. It must be shortened ·007 inch.

10. 32·25.   11. 93 approx.   12. 45 approx.

13. 5867 ft. nearly.   14. 5·4 approx.   15. 981.

16 32·12 at the equator; 32·28 at the pole.   18. 55.

19. $\frac{\pi}{4}$.   20. 8643 : 8639.   21. 4890 ft. nearly.

22. It must be shortened by ·0018 in. approx.

23. $\frac{2200 . n}{9}$ ft.   24. 432.   25. 39980.

XIX. (Page 242.)

1. $\left(\frac{m_1-m_2}{m_1+m_2}\right)^2 g$.   3. (1) 14 ft. per sec.; (2) 6 ft. per sec.

5. It moves with an acceleration $\frac{m_2\sqrt{m_1^2+m_2^2}}{(m_1+m_2)^2}g$, at an angle $\tan^{-1}\left(\frac{m_1}{m_2}\right)$ with the vertical.

6. $3200(\sqrt{2}+\sqrt{3})+896$ ft.

8. $4\sqrt{5}$ ft.-sec. units, at $\angle\tan^{-1}\frac{1}{2}$ with the horizon; a straight line.

9. 5·60 ft.; 29·32 ft.   14. $23y+7x=0$.

XX. (Page 252.)

1.	88 ft. per sec.	2.	$3\frac{1}{3}$ miles.	3.	18 ft.
4.	2 secs.	5.	352 ft.	6.	$\frac{3}{4}$ ft.-sec. units.
7.	$\frac{n^2f}{m}$.	8.	32 ft.	9.	20 secs.
10.	$130\frac{10}{11}$.	11.	$4\frac{1}{2}$ secs.	12.	11 ft.
13.	8.	14.	$9\frac{1}{3}$.	15.	8 lbs.
16.	1 sec., 8 ft.	17.	$5\frac{5}{11}$ miles.	18.	$\frac{11}{15}$.
19.	40 ft. per sec.	20.	30 secs.; 7200 ft.; 250 lbs.		
22.	15 secs.	24.	204,800.	25.	962,361,000.

XXI. (Page 259.)

1. $5\frac{1}{4}$ ozs. wt. 3. The pressure is diminished by one-seventh.

4. $17\frac{7}{9}$ units of impulse; $10\frac{2}{3}$ ft.-sec. units.

6. $mg\sqrt{3}$, $2mg$.

9. If the particle be of mass m lbs. the tension will be increased by $15mg$ poundals.

11. $\frac{4mv^2}{3a}$.

XXII. (Page 266.)

1. $\frac{3t^2}{2}$. 2. $3+2t^2$. 3. $\frac{5t^2}{2}$. 4. 90 ft.

5. 54 ft., $12t$. 6. $4t+2t^3$, $12t$. 7. $2+18t$, t^2+3t^3.

8. $20t$, at an angle $\tan^{-1}\frac{4}{3}$ with OX. Acceleration $=20$.

9. Acceleration $=10$, making an angle $\tan^{-1}\frac{4}{3}$ with OX.
Velocity $=5\,(1+2t)$.

MISCELLANEOUS EXAMPLES. XXIII. (Page 268.)

3. $\frac{u^2}{2\mu g}$. 6. $\frac{1056\sqrt{2}}{5}$ ($=298\cdot68$ approx.) ft.

10. The length begins at a point 162 ft. from the starting point.

11. $2\frac{1}{4}$ ozs. wt.; $3\frac{1}{2}$ ozs. wt. 12. $\frac{7u}{27}$; $\frac{8u}{27}$; $\frac{4u}{9}$.

13. (160, 36). 15. $1\frac{9}{16}$ inches.

16. 15586 ft. or 39 ft. approx. 17. 6 secs.

18. 120 ft. per sec. at an elevation $\tan^{-1}\left(\frac{3}{4}\right)$.

20. $v^2 \cos 2a = 4gr$.

23. $\frac{60}{\pi}$ (=19 nearly) revolutions; $3\frac{1}{8}$ lbs. wt.; $1\frac{7}{8}$ lbs wt.

24. $m = e^3$.

25. He ascends with a uniform acceleration $\frac{2h}{t^2}$ and his weight is $\frac{Mgt^2}{2h+gt^2}$ lbs. wt.

26. A straight line. 28. 36 approx.

29. Acceleration $= \frac{P-Q}{P+Q} g \cos\theta$; tension $= \frac{2PQg\cos\theta}{P+Q}$.

31. 1 foot. 32. 60 miles an hour. 33. $\frac{1}{24}$ ft.-sec. units.

34. Wind blows at an angle $\tan^{-1}\left(\frac{1-\cot a}{1-\tan\beta}\right)$ towards N. of W.; the smoke track would make an angle $\cot^{-1}\left(\frac{2-\tan\beta-\tan\beta\cot a}{1-\tan\beta}\right)$ N. of W.

36. $\mu g \cos a$. 38. $15\frac{5}{88}$ miles.

42. (1) g ft. per sec.; (2) the bodies move afterwards with a uniform velocity of $\frac{3g}{4}$ ft. per sec.

43. 16 ft.; 6 secs.

45. The speed of the wind is equal to that of the steamer, and it blows to the N.E.

47. To ensure motion μ must be less than $\frac{m \sin a \cos a}{M}$.

57. $\frac{4mMa}{4m^2 - M^2}$ ft. 60. $\sqrt{\frac{(r_2^2 - r_1^2)g}{4h}}$.

64. $g \tan a$. 68. $\sqrt{g(5a-3b)}$.

69. The beads exchange velocities at the first collision, and after the string tightens move with a common velocity $\frac{u}{2}$.

75. $\frac{44}{15}\left(\frac{f+f'}{ff'}\right)$ hours.

78. After $P-Q$ is attached, the bodies move with a uniform velocity $\frac{P-Q}{2P}gt$.

80. A straight line through the point of projection.

81. Velocity $= u \tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$ where u = velocity of projection; elevation $= \frac{\pi}{4}-\frac{\alpha}{2}$.

82. $\tan^{-1}\left(\sqrt{\frac{271}{90}}\right)$.

85. If α = angle of projection, and β = elevation of the plane, the time of flight $= \dfrac{2u\sin(\alpha-\beta)+\sqrt{4u^2\sin^2(\alpha-\beta)+8hg\cos^2\beta}}{2g\cos\beta}$.

88. $\dfrac{[m_1m_3-m_2m_3+\mu m_3(m_1+m_2)]g}{m_1+m_2+m_3}$; $\dfrac{(m_1m_3+2m_1m_2+\mu m_1m_3)g}{m_1+m_2+m_3}$.

90. $2\sqrt{\dfrac{2m+m'}{m'-m}ga}$. 99. $\dfrac{2g}{5}$.

104. The locus of G is a straight line through A; and its acceleration $= \dfrac{g\sqrt{(m_1\sin 2B-m_2\sin 2C)^2+(m_1\sin^2 B+m_2\sin^2 C)^2}}{m_1+m_2}$.

106. 1600 lbs. 110. $\tan^{-1}\left(\dfrac{2y-x\tan\alpha}{x}\right)$.

113. $\dfrac{a}{t\cos\phi}$; $\dfrac{1}{2}gt^2-a\tan\phi$. 114. 50 ft. per sec.; $\tan^{-1}\dfrac{3}{4}$.

115. $4\sqrt{3}+\dfrac{3}{4}$ (=7·68) lbs. wt.; $4-\dfrac{3\sqrt{3}}{4}$ (=2·7) lbs. wt.

117. 30 ft. per sec.; 10 ft. per sec. in opposite directions; 40 lbs. wt.

118. $2y=3(x+a)$; $y=3(x-a)$. 121. 3 : 8.

122. At an angle 45° with AB. 123. 960 ft.

125. $15w=W$.

127. Acceleration $= \dfrac{w\sin\alpha-w'\sin\alpha'}{w+w'}g$;

force $= w'\sin\alpha'-w\sin\alpha+\dfrac{ww'\sin(\alpha+\alpha')\sin(\alpha-\alpha')}{w+w'}$.

133. 1344 ft. per sec.; 1200 tons wt.
$11\frac{101}{113}$ ft. per sec.; 12 ft. per sec.

135. The descending mass 3 lbs. strikes the plane with the velocity $8\sqrt{\frac{3}{7}}$, leaves it with velocity $\frac{4}{7}\sqrt{\frac{6g}{7}}$ ft. per sec., rises to height $\frac{48}{49}$ ft.; again strikes the plane with velocity $\frac{32}{7}\sqrt{\frac{3}{7}}$ ft. per sec., and leaves it with $\frac{4}{7}$ths of this velocity, and so on.

138. θ must lie between 15° and 75°.

139. If the ball be thrown from a point distant b from one wall to strike the opposite wall at horizontal distance a, and if u be the velocity and α the angle of projection, e the coefficient of elasticity, the required height $= eu \sin \alpha t - \frac{1}{2}gt^2$, where $t = \dfrac{g(a+b+ae) - 2eu^2 \sin\alpha\cos\alpha}{eu\cos\alpha g}$.

EXAMINATION PAPERS.

Paper I. (Page 287.)

1. 30 : 11. 2. $\frac{4}{9}$ secs. 3. 635 lbs. wt.

4. 1600 ft.; $32\sqrt{101}$ $(=321{\cdot}6)$ ft. per sec.

6. $14\frac{2}{3}$ radians per sec.

7. If the mark be at a horizontal distance h, and a vertical distance k, from the point of projection, the initial velocity $= \sqrt{g\dfrac{h^2+4k^2}{2k}}$, and the angle of projection $= \tan^{-1}\left(\dfrac{2k}{h}\right)$.

Paper II. (Page 288.)

1. $4\frac{115}{128} = 4{\cdot}90$ tons wt. 2. The wt. of $\frac{2m}{3}$ lbs.

3. In 9 secs.

4. $\sqrt{u^2+v^2+2ft(u\cos\alpha+v\sin\alpha)+f^2t^2}$, at an angle $\tan^{-1}\left(\dfrac{v+f\sin\alpha t}{u+f\cos\alpha t}\right)$ with the former.

6. a.

Paper III. (Page 288.)

1. (1) 12 stone wt.; (2) 15 stone wt. 2. $W \cos a$.

3. $46\frac{7}{8}$ secs. 4. $2n-1:1$. 5. East; $v\sqrt{2}$.

6. $\dfrac{au}{\sqrt{v^2-u^2}}$.

Paper IV. (Page 289.)

1. $\dfrac{v}{3g}$ secs. 2. $27\frac{1}{2}$ secs. 3. $17\frac{1}{3}$ ft.

5. $e=\dfrac{3}{4}$. Impulse $=\dfrac{35}{4}\sqrt{5}$ $(=19{\cdot}57)$ units of impulse; $71\frac{3}{7}$ ft.

6. 2200 H.P.

Paper V. (Page 290.)

1. 9206·65 approx. 2. $AP=AB$.

3. $bl \sin a \operatorname{cosec} \beta \cos^2(a+\beta)$.

6. $\dfrac{2v^2\sin^2 a}{g}$, $\dfrac{2v^2\sin^2 a}{g}(e+2)^2$, $\dfrac{2v^2\sin^2 a}{g}(e^2+2e+2)^2$,

$$\frac{2v^2\sin^2 a}{g}(e^3+2e^2+2e+2)^2;$$

the latus rectum of the n^{th} path $=\dfrac{2v^2\sin^2 a}{g}\left(\dfrac{e^n+e^{n-1}-2}{e-1}\right)^2$.

Paper VI. (Page 290.)

2. $\dfrac{1}{2}$. 6. $\dfrac{g}{2}$ vertically downwards; 2 lbs. wt.

Paper VII. (Page 291.)

2. 9·06 lbs. wt. per ton. 3. $7\frac{3}{169}$ ft. per sec.; 5 : 784.

4. $\tan^{-1}(2\tan\beta)$, where β is the elevation of the given plane.

Paper VIII. (Page 292.)

1. The space begins 5 ft. from the top. 2. $20\frac{1}{8}$ ft.

3. At a height from the plane which $=\dfrac{10h_1h_2-h_1^2-9h_2^2}{16h_2}$.

4. $\dfrac{1}{4}$ sec. 4. $\dfrac{2u\tan\theta\cos a}{g}$.

PAPER IX. (PAGE 292.)

1. 19400 lbs. wt. 2. 1608 ft. per second.

6. By ·00012 of its length.

PAPER X. (PAGE 293.)

2. 1173 ft. per sec. 3. $62\frac{1}{2}$ ft. 4. 7840 ft.

6. If O be the centre, and A the highest point of the given circle, the locus is a circle on OA as diameter.

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